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A

TREATISE

ON THE

CONIC SECTIONS.

IN FIVE BOOKS.

BY G. WALKER, F.R.S.
OF NOTTINGHAM.

LONDON:
PRINTED FOR CHARLES DILLY, IN THE POULTRY.
1794.

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TO

FRANCIS MASERES, Esq.

CURSITOR BARON OF THE EXCHEQUER,

AS A TRIBUTE OF ESTEEM FOR HIS CHARACTER AND LEARNING,

AND OF GRATITUDE FOR PERSONAL KINDNESSES,

THIS WORK IS RESPECTFULLY DEDICATED,

BY HIS OBLIGED

AND OBEDIENT SERVANT,

G. WALKER.

PREFACE.

THE work, which is here offered to the public, is the fruit of many years agreeable attention, to which the Author was led by a persuasion that so simple a property as is exhibited in the 24. PROP. of the Universal Arithmetic, might contain the elements of all the properties of the three Conic Sections, and perhaps unfold them with more ease and elegance, than had been obtained by any other method. It is now near thirty years since he first discovered the property of the Generating Circle, but though it be an immediate consequence of the primary proposition, yet it was for some time hid from his view, nor does it appear that any Geometer had discovered it, though the property in Sir Isaac Newton's Arithmetic has so long been known. This discovery conducted the author with order and facility to every principal property of the Conic Sections. He observed in his progress that these Sections have more connection

with

with the Circle than with the Cone, nor is it any thing wonderful that it ſhould be ſo, ſince the Circle is the principal element from which the Cone itſelf is generated. The circle being therefore the common geneſis of the three ſections, the properties which are common to them all are deduced from the common ſource in one common demonſtration, to a much greater extent than in any treatiſe which deduces the ſection from the Cone; while the diſcrimination of each of the ſections, being a ſimple variation of the common geneſis, not only ſuggeſts the diſcriminating properties, but gratifies the underſtanding with the immediate perception of the operating cauſe of them. Indeed the affinity to and dependance of the ſections upon the circle is ſo immediate, that the demonſtrator has often little more to do than to transfer the various properties of the circle to the correſponding properties of the ſections.

The Author has founded his treatiſe on the Prop. of Sir Iſaac Newton, becauſe he was willing to adhere to the order in which his own mind was conducted in the enquiry. But as the property of the generating circle appears ſo early as in the 3. Prop., it will give little trouble to the Teacher or Reader of this treatiſe, if he commence with the generating circle, making this the primary geneſis of the ſections, and adapting the primary definitions thereto. Or, if he

he prefer the Cone as the foundation of the work, he may commence with the 54. Prop., and adapting as before the definitions thereto, he may paſs from thence directly to the 1. Prop. in this book, and thence in order without any other derangement. And this will be effected without any alteration in the demonſtrative part, as the 54. Prop., has no dependance on any of the preceding propoſitions, but appeals only to the elements of Euclid. Dr. Hamilton, in his Conic Sections, has noticed this property, but his demonſtration depends upon the preceding part of his work, and therefore the Author derived no other benefit from him with reſpect to this property, than what the mere notice of the property conveyed, but for this benefit he makes reſpectful acknowledgment. This property is limited to a Right Cone, but it is probable that a ſimilar property belongs to the Scalene Cone, correſponding to that extended property of the generating circle, which is delivered in the 52. Prop.

Whatever be the merit of the work, it is in the conſtruction and demonſtration almoſt wholly original, as the new principle, from which the whole is deduced, and which is kept in view throughout almoſt the whole of the I. Book, rendered the demonſtration of other Authors of no uſe in this Treatiſe. It was therefore from no pride or affectation that he declined to borrow, but from the

nature of his plan. Those Authors to whom in this Book he is alone indebted, are the late Professor Simpson of Glasgow, whose friendly notice the Author enjoyed while he was a Student in that University, Dr. Hamilton, and in one Prop. Sir Isaac Newton. Many new properties of the Sections, at least which the Author believes to be new, are dispersed through this first book, of which that of the generating circle is the principal. In the subsequent books, and particularly in the V., which treats of the Loci, more of novelty as to the properties of the Sections will be delivered. The I. Book alone is now published, because the others are not ready for publication; but as the principal properties both of the Sections in common, and of each in particular, are demonstrated in this book, it may be considered as in some degree compleat in itself. The materials of the other four books are mostly prepared, but it will require time to arrange and dispose the whole for the press. To do this the Author pledges himself to the public, if life and health be continued to him, and at this moment it would in a great degree have been done, if an unpleasant circumstance had not demanded his attention for a while to a very different object.

The Author has prefixed a few necessary Lemmata. This is reprehended as a blemish by the writer of the preface to Dr. Hamilton's Conic Sections. But these

Lemmata

Lemmata are general properties in pure Geometry, frequently appealed to in the treatiſe, and if not previouſly eſtabliſhed, the demonſtration of the propoſitions would have been burthened with matter that was not proper to them, and which muſt have been repeated even to diſguſt. The truth is, the Elements of Euclid are exceedingly defective. The doctrine of the harmonic ſection of a right line particularly requires to be added to the Elements of Geometry.

The Author has no profit in view, he never expected it from any of his former publications, and could not therefore expect it in a work of abſtract ſcience. It is however unpleaſant and inconvenient to him to be ſubjected to actual loſs, which he fears he muſt encounter to a conſiderable extent. From this he hoped to have been relieved by the generoſity of the Cambridge Preſs, with the Directors of which his Manuſcript was depoſited for ſome conſiderable time in the early part of the year 1793. Encouraged to hope for this favour, the encouragement and hope failed him, the manuſcript was returned, and the favour was not extended. As this Preſs is profeſſedly for the encouragement of ſcience, the Public muſt judge whether the work which invited its patronage, was deſerving of it or no. Perhaps it was the misfortune of the Author to be a Diſſenter, when it has become

the temper and very principle of the day to cut off a Diſſenter from every public expectation. But ſurely, however wiſe the general interdict may be, pure innocent ſcience might have promiſed itſelf an exemption from the malediction both of religious and political party.

ERRATA.

Page	Line	
18.	29.	dele *ſupple.*
37.	26.	*for* the ſecond A *r.* A.
46.	27.	in the margin, *for* xii. *r.* n.
47.	10.	in the margin, *for* II. *r.* n.
78.	26.	*for* on, *r.* or.
100.	24.	*for* (1. COR. DEF.) *r.* (16. DEF.)
104.	32.	*for* FE *r.* FE.

SUB-

ON THE

PROPERTIES

COMMON TO THE

THREE SECTIONS.

LIB. I.

LEMMATA

TO THE

CONIC SECTIONS.

LEM. I.

IF three right lines be parallel to each other, or meet in one common point, and from two points in one of them, right lines parallel to each other be drawn to each of the other two, the parallels meeting one of these lines shall be proportional to the parallels meeting the other. LEM. I.

FIG. 1. 2. 3. 4.

Let AB, AC, AD be parallel to each other, or meet in one common point A, and from two points C, c, in one of them AC, be drawn to AB the parallels CB, cb, and to AD the parallels CD, cd; I say that CB shall be to cb as CD is to cd.

If AB, AC, AD, be mutually parallel, CB will be equal to cb, and CD to cd, therefore CB is to cb, as CD is to cd.

But

FIG. But if AB, AC, AD, meet in one common point A, then because of the parallels CB, cb, and CD, cd, CB will be to cb (as AC is to Ac, viz.) as CD is to cd (4. e. 6.)

LEM. 2.

If from two of the angles of a triangle right lines be drawn perpendicular to the opposite sides, the right line joining their concourse and the remaining angle shall be perpendicular to the remaining side.

5, 6. Let ABC be a triangle, and from two of its Angles, A, C, be drawn AE, CD, perpendicular to the opposite sides BC, AB, meeting them in E, D, and meeting each other in G. I say that BG, being joined, shall be perpendicular to the remaining side AC.

Let BG meet AC in F, and join DE. Because the angles at D, E are right, the four points B, E, G, D, are in a circle, as also the four points A, C, E, D. Therefore in the first circle, the angle DBG or ABF is equal to the angle DEG, and in the second circle the angle ACD is equal to the angle DEG (21. & 22. e. 3.) Wherefore the angle ABF is equal to the angle ACD, and consequently the four points B, C, F, D, are in a circle. The angle BDC is therefore equal to the angle BFC, and the angle BDC being right, the other BFC will be right also.

COR. The three right lines drawn from the three angles of a triangle perpendicular to the opposite sides meet in one common point.

DEFINITIONS.

1. If in a right line given in position four points be assumed, such that the whole be to one extreme segment as the other extreme segment is to the intermediate part, the right line is said to be divided HARMONICALLY in these four points, and the points themselves are called HARMONIC POINTS.

Or,

Or, If in a right line given in position and magnitude two points be assumed, the one within, the other without the primary terms of the right line, and the distances of each of these points from the terms be to each other in the same ratio, the right line is said to be HARMONICALLY divided, and the point within the terms is called the INTERNAL, the other without is called the EXTERNAL harmonic point. FIG.

Thus if in AB, given in position and magnitude, two points, D, d, be assumed, the one D within, the other d without the terms A, B; and such, that AD be to BD as Ad is to Bd; the right line AB is said to be harmonically divided in the points D, d, and the one D is called the internal, the other d the external harmonic point. 7, 8.

NOTE. These are only different modes of defining the same thing. For the two terms A, B, together with D, d, make up the four harmonic points of the first definition; and the whole Ad is to the extreme segment Bd as the other extreme segment AD is to the intermediate part BD.

Also, if Dd were given, A, B, would be the two points, in which Dd is harmonically divided.

2. If from the four points in which a right line is harmonically divided right lines be inflected to any point without, these four right lines are called HARMONICALS.

LEM. 3. PROBLEM.

A right line being given in position and magnitude, it is required to find a point therein, whose distances from the terms of the given right line shall be in a given ratio, viz. in the ratio of two right lines given in magnitude.

 Let

FIG. But if AB, AC, AD, meet in one common point A, then because of the parallels CB, cb, and CD, cd, CB will be to cb (as AC is to Ac, viz.) as CD is to cd (4. e. 6.)

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Let BG meet AC in F, and join DE. Because the angles at D, E are right, the four points, B, E, G, D, are in a circle, as also the four points A, C, E, D. Therefore in the first circle, the angle DBG or ABF is equal to the angle DEG, and in the second circle the angle ACD is equal to the angle DEG (21. & 22. e. 3.) Wherefore the angle ABF is equal to the angle ACD, and consequently the four points B, C, F, D, are in a circle. The angle BDC is therefore equal to the angle BFC, and the angle BDC being right, the other BFC will be right also.

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Thus if in AB, given in position and magnitude, two points, D, d, be assumed, the one D within, the other d without the terms A, B; and such, that AD be to BD as Ad is to Bd; the right line AB is said to be harmonically divided in the points D, d, and the one D is called the internal, the other d the external harmonic point. 7, 8.

NOTE. These are only different modes of defining the same thing. For the two terms A, B, together with D, d, make up the four harmonic points of the first definition; and the whole Ad is to the extreme segment Bd as the other extreme segment AD is to the intermediate part BD.

Also, if Dd were given, A, B, would be the two points, in which Dd is harmonically divided.

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LEM. 3. PROBLEM.

A right line being given in position and magnitude, it is required to find a point therein, whose distances from the terms of the given right line shall be in a given ratio, viz. in the ratio of two right lines given in magnitude.

FIG. 7, 8, 9. Let AB be the right line given in position and magnitude, F, G, the two right lines given in magnitude, and suppose the thing to be done, viz. that a point D is found in AB, such that AD is to BD, as F is to G.

Draw any right line through A, in which take AC equal to F. Join DC, and draw BE parallel to DC, meeting AC in E. Because of the parallels, AD is to BD as AC or F is to CE, and by supposition, AD is to BD as F is to G. Therefore CE is equal to G, and will be given in magnitude. But the point C and the position of CE are given, therefore the point E and the position of BE are given. Wherefore CD being parallel to BD, and the point C being given, CD and the point D will be given.

To the composition of this problem it will be required, that in the right line AC, and from the point C, be placed CE equal to G. But the point E may be situated towards either part of C, viz. two points E, E, each at the distance of G from C, equally answer to the analysis; and therefore if the given ratio be that of inequality,
7, 8. two points D, D, will be found to answer the conditions of the problem; and these two points will be towards the same parts of A with B, if F be greater than G, and E, E, fall upon the same side of A with C; but towards different parts of A, if F be less than G, and E, E, fall in AC towards different parts of A. But
9. if the given ratio be that of equality, viz. if F be equal to G, CE then becomes equal to CA, and the points E, A, coinciding, BE becomes BA itself. Wherefore CD being parallel to BE or BA, the point D vanishes, and the problem admits of only one solution in the single point D, answering to the single point E.

COMPOSITION. Through A draw any right line, in which take AC equal to F, and in the same, towards both parts of C, take CE, CE, each equal to G, if F be unequal to G; but if it be equal thereto, on the part of C opposite to A make CE equal to G. Join BE, BE, and draw CD, CD parallel to BE, BE. The points D, D, when the ratio is that of inequality, otherwise the point D alone, will answer to the conditions of the problem.

 For,

FIG.

For, because of the parallels, AD is to BD, and also Ad is to Bd, (as AC is to CE or Ce, viz.) as F is to G.

COR. 1. If the given ratio be that of a greater to a less, the points D, d, fall towards the same parts of A with B, viz. one between the terms A, B, the other without in AB produced, and on the parts of B opposite to A. If the given ratio be that of a less to a greater, the points D, d, fall on different sides of A, viz. the one between the terms A, B, the other in BA produced, on the parts of A opposite to B. But if the ratio be that of equality, the single point D is that in which AB is bisected.

COR. 2. AB is harmonically divided in D, d.—For Ad is to Bd (as F is to G, viz.) as AD is to BD.

COR. 3. If two right lines meet in A, and from a point C in one of them be drawn CD, Cd to the other, and from a point B in the other be drawn BE, Be, parallel to CD, Cd, and meet AC in E, e; then if Ee be bisected in C, AB shall be harmonically divided in D, d; and conversely, if AB be harmonically divided in D, d, Ee shall be bisected in C.

COR. 4. If a right line be harmonically divided, no third point can be assumed in the right line, so that in this point together with either of the two former harmonic points the same right line shall be again harmonically divided.

If AB be harmonically divided in D, d, no other point F can be assumed in AB, so that AD shall be to BD as AF is to BF. For if possible, then because AD is to BD also as Ad is to Bd, therefore ex æquo, Ad is to Bd as AF is to BF, and dividendo or componendo, Ad is to AB as AF is to AB. Wherefore Ad is equal to AF, a part to the whole, which is absurd. 7.

LEM.

LEM. 4.

FIG. **If four right lines be harmonicals, a right line parallel to one of them, and falling upon the other three, ſhall be biſected in the middle concourſe.**

And converſely, if four right lines meet in a common point, and a right line parallel to one of them, and falling upon the other three, be biſected in the middle concourſe, theſe four right lines ſhall be harmonicals.

10. Let AB be a right line harmonically divided in D, d, and AE, DE, BE, dE, be four harmonicals inflected from A, D, B, d, to a common point E. I ſay, that a right line parallel to any one of the four, as dE, and falling upon the other three, ſhall be biſected in the middle concourſe.

Firſt, let the parallel be drawn through either of the harmonic points, as D, and meet AE, BE in L, I. I ſay that IL is biſected in D. Becauſe AB is harmonically divided in D, d; Ad is to AD as Bd is to BD, and on account of the parallels, Ad is to AD as Ed is to DL, and Bd is to BD as Ed is to DI. Therefore ex æquo, Ed is to DL as Ed is to DI. Wherefore DL is equal to DI.

Now let any right line parallel to dE, meet AE, DE, BE, in G, H, F. I ſay that it ſhall be biſected in the middle concourſe H. The ſame things remaining, FG is divided in H in the ſame proportion as IL is divided in D (LEM. 1.). Therefore IL being biſected in D, FG is biſected in H.

Converſely, if four right lines EP, EQ, ER, ES, meet in a common point E, and FG parallel to one of them, as ES, meet the other three, and be biſected in the middle concourſe H; the four right lines EP, EQ, ER, ES, ſhall be harmonicals.

Draw

Draw any right line falling upon them in the points A, D, B, D, FIG.
and through D the concourse with the same right line which meets FG in H, draw IDL parallel to ES or FG, meeting ER, EP, in I, L. Because FG is bisected in H, IL will be bisected in D. But on account of the parallels, AD is to AD as ED is to DL or ID, and BD is to BD as ED is to ID. Therefore ex æquo, AD is to AD as BD is to BD, viz. AB is harmonically divided in D, D (1. DEF.), and consequently EA, ED, EB, ED, or EP, EQ, ER, ES, are harmonicals (2. DEF.).

COR. 1. Every right line falling upon four harmonicals, is harmonically divided in the four points in which it meets them.

For, being harmonicals, a right line parallel to one of them, and falling upon the other three, will be bisected in the middle concourse; and therefore, as has appeared in the demonstration of the converse, every right line falling upon the four will be harmonically divided in its concourses with them.

COR. 2. If there be four harmonicals, and through any point be drawn four right lines parallel to them, these four right lines shall also be harmonicals.

Let EP, EQ, ER, ES, be harmonicals, and through any point 10, 11.
E be drawn EP, EQ, ER, ES, parallel to EP, EQ, ER, ES; I say, that EP, EQ, ER, ES, are also harmonicals.

Draw any right line FHG parallel to ES, meeting ER, EQ, EP, in F, H, G, and also FHG parallel to ES, meeting ER, EQ, EP, in F, H, G. Because ES is parallel to ES, FHG will be parallel to FHG. Therefore the triangles EFH, EHG, will be equiangular to the triangles EFH, EHG, and consequently FG will be divided in H in the same proportion as FG is divided in H. But FG is bisected in H, therefore FG is also bisected in H, and consequently EP, EQ, ER, ES, are also harmonicals.

COR. 3. If through the vertex of a triangle two right lines be drawn, one parallel to the base, the other bisecting the base, these two together with the sides of the triangle shall be harmonicals.

Let

FIG. 10. Let EFG be any triangle, and through the vertex E be drawn ES parallel to the base FG, and EH bisecting FG in H; I say, that ES, EF, EH, EG, shall be harmonicals.

For ES, EF, EH, EG, are four right lines meeting in a common point E, and a right line FHG parallel to one of them ES, and meeting the other three in F, H, G, is bisected in the middle concourse H. Therefore ES, EF, EH, EG, are harmonicals.

COR. 4. If there be a parallelogram, whose diagonals are drawn, and through one of the angles of the parallelogram a parallel to the opposite diagonal be drawn; this parallel, the diagonal drawn from the same angle, and the two sides of the parallelogram about the same angle, shall be harmonicals.

12. Let ABCD be a parallelogram, AC, BD, its diagonals. If AE be drawn parallel to BD; I say, that AE, AD, AC, AB, shall be harmonicals.

Let AE meet CD in E. Because ABCD and ABDE are parallelograms, CD, DE are each equal to AB, and therefore equal to each other. Wherefore in the triangle ACE, AB, AD are drawn, the one parallel to the base CE, the other to bisect the base CE, and consequently AB, AC, AD, AE, are harmonicals.

LEM. 5.

If a right line be harmonically divided, and be also bisected, the rectangle under the segment intercepted between the two points of harmonic section, and the segment intercepted between either of these points and the point of bisection, shall be equal to the rectangle under the two segments intercepted between the same harmonic point and the terms of the right line.

And

And conversely, if a right line be bisected, and two other points be assumed therein, one within the terms of the right line, the other without in the right line produced, and if the rectangle under the segment intercepted between the points assumed, and the segment intercepted between either of these points and the point of bisection, be equal to the rectangle under the segments intercepted between the same point and the terms of the right line; the right line shall be harmonically divided in the points assumed. FIG.

Let AB be harmonically divided in D, d, and bisected in C; I say, that the rectangle DdC shall be equal to the rectangle AdB. 13, 14.

At right angles to AB draw AF equal to AB, and joining BF, draw dE parallel to AF meeting BF in E. Join AE, and DE meeting AF in G, also compleat the rectangle AGLD. Draw EP parallel to AB, meeting AF, LD in P, C, and let GL meet EF, Ed in H, I.—Because AB is harmonically divided in D, d, AE, DE, BE, dE are harmonicals, and therefore AF parallel to one of them dE, and meeting the other three in F, G, A, will be bisected in the middle concourse G (LEM. 4.). But AB, equal to AF, is also bisected in C, therefore BC is equal to AG, viz. to dI; and because AB is equal to AF, Bd will be equal to Ed, and the whole or remainder dC will be equal to the whole or remainder EI. Consequently the rectangle IO, being contained under EI, EO, is the same with the rectangle DdC; and the rectangle Pd, being contained under Ad, dE, is the same with the rectangle AdB. But because the rectangles Od, PI, stand about the diameter EG, their complements IO, Pd, will be equal between themselves (43. e. 1.). Therefore the rectangle DdC will be equal to the rectangle AdB.

FIG. Converſely, If a right line AB be biſected in C, and two points D, D, be aſſumed therein, one within, the other without in AB produced, and the rectangle DDC be equal to the rectangle ADB, the right line AB ſhall be harmonically divided in D, D.

The ſame conſtruction remaining, becauſe the complements IO, PD, are equal, and the complement PD is the ſame with the rectangle ADB, and is therefore equal to the rectangle DDC, the rectangle IO will be equal to the rectangle DDC. But EO, one of the ſides under which the rectangle IO is contained, is equal to DD; therefore the other ſide EI will be equal to DC. Alſo, for the ſame reaſon as in the preceding, ED is equal to DB, therefore the whole or the remainder DI, or its equal AG, will be equal to the whole or the remainder BC. But AB is biſected in C, therefore AF, which is equal to AB, will alſo be biſected in G. Wherefore becauſe the four right lines AE, DE, BE, DE, meet in a common point E, and the right line AF parallel to one of them DE, is biſected in the middle concourſe G, theſe four right lines will be harmonicals (4. LEM.), and AB falling upon them in A, D, B, D, will be harmonically divided (1. COR. LEM. 4.).

COR. 1. The diſtances of either harmonic point from either term of the right line and from the point of biſection are in the proportion of the diſtance of the ſame term from the other harmonic point to half the line.

14. The ſame things remaining, I ſay, that AD is to CD as AD is to AC; or that BD is to CD as BD is to BC.—For the rectangle ADB being equal to the rectangle CDD, AD is to CD as DD is to DB. Wherefore dividendo, AD is to CD as AD is to BC or AC.—Alſo becauſe BD is to CD as DD is to AD, therefore dividendo, BD is to CD as BD is to AC or BC.

COR. 2. If a right line be harmonically divided, and alſo biſected, half the line ſhall be a mean proportional between the diſtances of the point of biſection from the harmonic points.

The ſame things remaining, I ſay, that AC or BC ſhall be a mean proportional between CD, CD.

By

By the preceding Cor. Ad is to AD as Cd is to AC, and also Ad is to AD as BC or AC is to CD. Therefore ex æquo, Cd is to AC as AC is to CD. FIG.

LEM. 6.

If a right line be harmonically divided, and from the terms of the right line two right lines be inflected to a point without the same, at right angles to each other, the right lines inflected from the two harmonic points to the same point without, shall make equal angles with each of the lines inflected at right angles.

And conversely, if from the terms of a right line two right lines be inflected to a point without, at right angles to each other, and from two points assumed in the right line, one within, the other without in the right line produced, if two right lines be inflected to the same point as the former, and make equal angles with them, the right line shall be harmonically divided in the two assumed points.

Let AB be harmonically divided in D, d, and from A, B, be inflected AE, BE to a point E at right angles to each other; I say, that DE, dE, being joined, shall make equal angles with AE, BE. 15.

Through B draw FBG parallel to AE, and meeting DE, dE, in F, G. Because AE is perpendicular to BE, FG will also be perpendicular to BE. And AB being harmonically divided in D, d, AE, DE, BE, dE, are harmonicals, and therefore FG, which is parallel to AE, will be bisected in B (4. LEM.). Wherefore FB

 being

FIG. being equal to BG, BE common, and the angles at B each right, the angle FEB will be equal to the angle GEB, and the angle EFB to the angle EGB (4. E. 1.); viz. because of the parallels, the angle FEA is equal to the angle GEH. Therefore DE, dE, make equal angles, internally with BE, and externally with AE.

And conversely, if in a right line AB two points D, d, be assumed, the one D within, the other d without in AB produced, and such, that AE, BE, being inflected to a point E at right angles to each other, if DE, dE, inflected to the same point E make equal angles with AE, BE; I say, that AB shall be harmonically divided in Dd.

Because DE, dE, make equal angles internally with one, and externally with the other of the two right lines BE, AE, therefore BD is to Bd as DE is to dE, and AD is to Ad as DE is to dE, (3. e. 6.). Wherefore ex æquo, AD is to Ad as BD is to Bd, viz. AB is harmonically divided in D, d.

COR. 1. Also conversely, if a right line AB be harmonically divided in Dd, and AE, DE, BE, dE, being inflected to a point E, if DE, dE, make equal angles with either AE, or BE; I say, that AE, BE, shall be at right angles to each other.

Let DE, dE make equal angles with BE. Therefore DE is to dE (as BD is to Bd (3. e. 6.) viz. on account of the harmonic section of AB) as AD is to Ad. Wherefore DE, dE, make equal angles also with AE (3. e. 6.), and consequently the angle AEB is a right angle.

COR. 2. If from two given points as D, d, be inflected two right lines DE, dE, to a point E, and in a given ratio, which is not that of equality, the LOCUS of the point E shall be a CIRCLE, given in position and magnitude.

Join Dd, and since the ratio is not that of equality, find two points therein, viz. A, B, such that AD may be to Ad, and BD be to Bd in the given ratio (LEM. 3.). Join AE, BE. Then AB is harmonically divided in D, d, and AE, DE, BE, dE, are harmonicals. Also because BD is to Bd as DE is to dE, the right

right lines DE, dE make equal angles with BE, and consequently FIG.
AE, BE are at right angles to each other (by the preceding COR.), and the point E is in a semicircle whose diameter is AB (CONVERSE. 31. e. 3.). But because the points D, d, and the ratio of BD to Bd, and of AD to Ad, are given, the points A, B, and the circle, which is the locus of the point E, will be given.

COR. 3. Hence if AB the diameter of a circle be harmonically divided in D, d, and from D, d, be inflected DE, dE, to any point E in the circumference, DE shall be to dE in the constant ratio of BD to Bd, or of AD to Ad.

For by the preceding COR. the locus of every point E, whose property is that DE is to dE, as BD is to Bd, and as AD is to Ad, is the circumference of a circle, whose diameter is AB. But E is in the circumference of this circle, therefore DE is to dE as BD is to Bd, or AD to Ad.

LEM. 7.

A diameter of a circle perpendicular to a right line in the same bisects each circumference subtended by the right line. And conversely, if the circumferences subtended by a right line in the same be bisected by a diameter, the right line and the diameter shall be at right angles to each other.

Let ADE be a circle, DE a right line in the same, and AB the diameter perpendicular to DE. I say, that the circumference EAD is bisected in A, and the circumference EBD in B. 16.

Let AB meet ED in F, and join AD, AE. Because AB is perpendicular to DE, DF will be equal to FE (3. e. 3.), while AF is common, and the angles at F are equal, because each right, therefore the angle DAF is equal to the angle EAF (4. e. 1.);

viz.

FIG. viz. the angle DAB is equal to the angle EAB, and consequently the circumference DB is equal to the circumference EB (26. e. 3.). And in the same manner it may be shewn that the circumference AD is equal to the circumference AE.

And conversely, if the diameter AB bisect the circumferences DAE, DBE, then shall AB be perpendicular to DE.

The same things remaining, because of the equal circumferences BD, BE, and AD, AE, the angle BAD is equal to the angle BAE, and the angle ADE to the angle AED (27. e. 3.), viz. the angle DAF is equal to the angle EAF, and the angle ADF to the angle AEF. Wherefore the side AF being common, the side DF shall be equal to the side EF (26. e. 1.), and consequently AB be perpendicular to DE (3. e. 3.)

LEM. 8.

If two right lines touching a circle meet each other, they shall be equal between themselves, and the diameter drawn through their concourse shall be perpendicular to the right line joining the points of contact.

17. Let the right lines DF, DG, touching a circle in the points F, G, meet each other in D; I say that DF shall be equal to DG, and that the diameter AB which passes through the concourse D shall be perpendicular to FG, which joins the points of contact.

From the centre C draw CF, CG. Because the sides CF, CD, are respectively equal to the sides CG, CD, and the homologous angles CFD, CGD are equal, being each right (18. e. 3.), the side DF shall be equal to the side DG, and the angle FCD be equal to the angle GCD (26. e. 1.). Wherefore the circumference AF is equal to the circumference AG (26. e. 3.), and consequently AB is perpendicular to FG (LEM. 7.)

LEM.

LEM. 9.

If two right lines touching a circle meet each other, every right line drawn through the concourse to cut the circle shall be harmonically divided in the concourse of the tangents, in its concourse with the right line joining the points of contact, and in its two concourses with the circle. FIG.

CASE 1. When the right line drawn is the diameter itself. 17.

The same things remaining, I say that the diameter AB is harmonically divided in D, d.

Join AF, BF, AG, BG. Because the diameter AB passes through the concourse of the tangents, it is perpendicular to FG (LEM. 8.), and therefore bisects the circumferences FAG (LEM. 7.). Wherefore the angle AGF is equal to the angle AFG (27. e. 3.). But because dF touches the circle, the angle AFd is equal to the angle AGF (32. e. 3.), therefore the angle AFd is equal to the angle AFG. The two right lines dF, DF, do therefore make equal angles with AF, and AF, FB are at right angles to each other (31. e. 3.), and consequently AB is harmonically divided in D, d. (LEM. 6.)

CASE 2. When the right line drawn is not a diameter.

Let HI drawn through the concourse d cut the circle in H, I 18. and FG in E. I say, that HI shall be harmonically divided in d, E.

Every thing remaining the same as in CASE 1, join DH, DI. Because AB is harmonically divided in d, D, and from these points are inflected dH, DH to a point H in the circle, dH will be to DH as Ad to AD (3. COR. LEM. 6.). For the same reason is dI to DI as Ad to AD; therefore ex æquo, dH is to DH as dI to DI,

FIG. DI, and by alternation, dH is to dI as DH is to DI. Wherefore DH, DI, make equal angles with Dd (3. e. 6.), and ED, dD are at right angles to each other. The right line dE is therefore harmonically divided in d, H, E, I (LEM. 6.)

COR. 1. If a right line touching a circle meet a diameter of the same, and from the point of contact a right line be drawn perpendicular to the diameter, the touching line and the perpendicular line shall make equal angles with the right lines drawn from the extremity of the diameter to the point of contact.

17. For dG being drawn as in the Prop. then FG being joined will be the same with the perpendicular FD drawn from the point of contact F to the diameter AB, and therefore the Cor. includes what has been demonstrated in the Prop.

COR. 2. If a right line touching a circle meet a diameter, and from the point of contact be drawn a perpendicular to the same diameter, the diameter shall be harmonically divided in the two points of concourse with the touching line and the perpendicular.

By the same illustration as in the preceding Cor. this will appear to be the same as the Prop. itself.

LEM. 10.

If a right line cutting a circle be harmonically divided, and through either harmonic point a perpendicular line be drawn to the diameter passing through the other harmonic point, every right line drawn through this other point to cut the circle shall be harmonically divided in the same point, in its concourse with the perpendicular, and in its concourses with the circle.

CASE

Case 1. When the right line harmonically divided is a diameter of the circle. FIG.

Let AB be a diameter of a circle harmonically divided in D, d, and HI a right line which drawn through either harmonic point d, and cutting the circle in H, I, meets in E a perpendicular to AB drawn through the other harmonic point D; I say that HI shall be harmonically divided in H, I, d, E. 18, 19.

The demonstration of this is the same as in Case 2. of the preceding Prop.

Case 2. When the right line harmonically divided is not a diameter.

Let HI, not a diameter, cut the circle in H, I, and be harmonically divided in d, E. Through either point d draw the diameter AB, cutting the circle in A, B, and meeting in D a perpendicular to AB drawn through the other harmonic point E.—I say, first, that AB shall be harmonically divided in D, d. Join DI, DH, draw HG parallel to ED, meeting DI in G, AB in L, and join AG, AH, AI, BI. Because DE, DI, Dd, DH, are harmonicals, and HG is parallel to DE; GH will be bisected in L (Lem. 4.) But the point H is in the circle, and the diameter AB is perpendicular to HG, therefore the point G is in the circle also (3. e. 3.), and consequently the circumference HBG will be bisected in B (Lem. 7.) The angle BIG or BID is therefore equal to the angle BAH (27. e. 3.), and the angle BId is equal to the angle BAH (21 & 22. e. 3.) Wherefore the angle BID is equal to the angle BId, and the angle AIB is right (31. e. 3.), and consequently AB is harmonically divided in D, d (Lem. 6.) 20, 21.

Again, Let any right line PO be drawn through d, cutting the circle in P, O, and meeting in R a perpendicular to AB drawn through the other harmonic point E. I say, that if HI be harmonically divided in d, E, PO shall be harmonically divided in d, R. By the preceding case, AB is harmonically divided in d, D; therefore by the first case, PO will be harmonically divided in d, R. 22, 23.

FIG. COR. 1. If two right lines cutting a circle meet each other, and each being harmonically divided, if their concourse be one of the harmonic points, the right line joining the two other harmonic points shall be perpendicular to the diameter passing through the concourse; and therefore every right line drawn through the concourse to meet the right line joining the other harmonic points, and cut the circle, shall be harmonically divided in its concourses with the circle.

First, let one of the right lines be the diameter itself.

20, 21. Let a right line HI cutting a circle and harmonically divided in D, E, meet in the point D the diameter AB, which is harmonically divided in D, D. I say, that ED being joined shall be perpendicular to AB.—If not, draw a right line from D perpendicular to AB, and meeting HI in some other point than E, then in that other point and D would HI be again harmonically divided, which is absurd (4. COR. LEM. 3.) Therefore the perpendicular to AB at the point D passes through E.

Secondly, when neither of the right lines is a diameter.

22, 23. Let HI, PO, cutting a circle in H, I, and P, O, and harmonically divided, the one in D, E, the other in D, R, meet each other in D; I say that ER being joined shall be perpendicular to the diameter AB passing through D. If not, let a perpendicular to AB drawn from E pass through some other point than R, then in that other point and D would PO be again harmonically divided, which is absurd. Therefore the perpendicular to AB drawn from E must pass through R. Wherefore because DE or RE is perpendicular to the diameter drawn through D, every right line drawn through D to cut the circle and meet DE or RE, will, by this Prop. be harmonically divided in D, supple, in its concourses with the circle, and in its concourse with DE or RE.

COR. 2. If a right line cutting a circle be harmonically divided, and in the exterior harmonic point meet a right line touching the circle, the interior harmonic point and the point of contact shall be in

in a right line perpendicular to the diameter passing through the point of concourse. FIG.

First, let the right line harmonically divided be the diameter itself.

Let the diameter AB harmonically divided in D, D, meet in the exterior point D the right line DF touching the circle in F. I say, that DF, being joined, shall be perpendicular to AB. If not, yet if a right line be drawn from F at right angles to AB, AB shall be harmonically divided in the concourse and in D (2. COR. LEM. 9.) while it is also harmonically divided in D, D, which is absurd (4. COR. LEM. 3.) Therefore a right line drawn from F at right angles to AB passes through D. 17.

Secondly, let the right line not be the diameter.

Let HI harmonically divided in E, D, meet in the exterior harmonic point D the right line DF touching the circle in F, and the diameter ABD be drawn. I say that FE shall be perpendicular to AB. Draw ED perpendicular to AB, then AB is harmonically divided in D, D (by this LEM.) and therefore FD, being joined, will be perpendicular to AB (by the first part of this COR.) Wherefore the points F, E, are in the same right line perpendicular to AB.

COR. 3. Hence if any number of right lines cutting a circle and harmonically divided meet each other in a common harmonic point, and in the same point meet one or two right lines touching the same circle, the interior harmonic points in each and the points of contact shall be in a right line at right angles to the diameter which passes through the point of concourse.

COR. 4. If ABI be a circle, D a point in the plane of the circle, but being neither the centre, nor in the circumferenc, MN a right line in the same plane, which is not perpendicular to the diameter drawn through D; then if a right line HI drawn through D cut the circle in H, I, and meet MN in E, and be harmonically divided in D, E; I say, that no other right line can be drawn through D to cut the circle, and be harmonically divided in D, and in its concourse with MN. 22, 23.

FIG. If possible, let GK cutting the circle in G, K, MN in L, and passing through D, be harmonically divided in D, L. Through D draw the diameter AB, and draw ED, LQ perpendicular to AB. Then AB is harmonically divided in D, D, and also in D, Q, which is absurd (4. COR. LEM. 3.) Therefore no right line but HI can be drawn through D to cut the circle, which shall be harmonically divided in D and in its concourse with MN.

COR. 5. If two right lines touching a circle meet each other, and through the concourse be drawn any right line to meet the right line joining the points of contact; the diameter perpendicular to the right line drawn shall be harmonically divided in its concourse therewith, and in its concourse with the right line joining the points of contact.

Let the right lines IL, HL, touching a circle in the points I, H, meet each other in L, and IH being joined,

24. First, let the right line drawn from L fall without the circle, viz. let LE meet in the point E without the circle. I say, that the diameter AB perpendicular to LE, and meeting LE, IH in D, D, shall be harmonically divided in D, D.

Join CI. Because the angles LDC, LIC are right, and the angle CLD is greater than the angle CLI, the angle LCD will be less than the angle LCI, and therefore CD will meet IH between the terms I, H, viz. the point D will be within the circle. Join LD, meeting the circle in P, O. PO is harmonically divided in L, D (LEM. 9.) Wherefore LE being drawn through one of the harmonic points L, perpendicular to the diameter AB drawn through the other harmonic point D, AB will by this Prop. be harmonically divided in D, D.

Secondly, let the right line drawn from L meet IH in a point D within the circle; I say, that the diameter KG perpendicular to LD, and meeting LD in R, IH in E, shall be harmonically divided in R, E. Join CL meeting IH in Q, also LE, and CD meeting the circle in A, B, and LE in D. Because EQ is perpen-

 dicular

dicular to CL (LEM. 8.), and LR by construction is perpendicular to CE, therefore CD which is drawn through the concourse of EQ, LR, will be perpendicular to LE (LEM. 2.) Wherefore by this Prop. IH will be harmonically divided in E, D, and consequently KG will also be harmonically divided in E, R. FIG.

LEM. 11.

If a right line cutting a circle be harmonically divided, and do not pass through the centre, the right lines touching the circle in the points in which the harmonic line cuts the circle, shall meet in a right line given in position, viz. in the right line drawn through one of the harmonic points at right angles to the diameter drawn through the other harmonic point.

Let HI, not passing through the centre, cut a circle in H, I, and be harmonically divided in D, E. Draw HL, IL, touching the circle in H, I, and meeting each other in L; I say, that the point L shall be in a right line drawn through either of the harmonic points D, E, at right angles to the diameter drawn through the other harmonic point. 24.

Join LE, and D being the inner harmonic point, draw LD, meeting the circle in P, O; also join the diameters CE, CD, CL, meeting LD in R, LE in D, and HI in Q. Then PO is harmonically divided in L, D (LEM. 9.), and by hypothesis HI is harmonically divided in E, D; therefore LE is perpendicular to CD (1. COR. LEM. 9.), and CL is perpendicular to HI (LEM. 8.) Wherefore in the triangle LDE, because DD, LQ, drawn from the angles D, L, perpendicular to the opposite sides LE, ED, meet in C, the right line EC joining the remaining angle E and the

FIG. the concourse C shall be perpendicular to the remaining side L D (LEM. 2.) The point L therefore in which the tangents HL, IL, meet, is in the right line LE drawn through the harmonic point E at right angles to the diameter C D drawn through the other harmonic point D, and it is also in the right line L D drawn through the harmonic point D at right angles to the diameter drawn through the other harmonic point E.

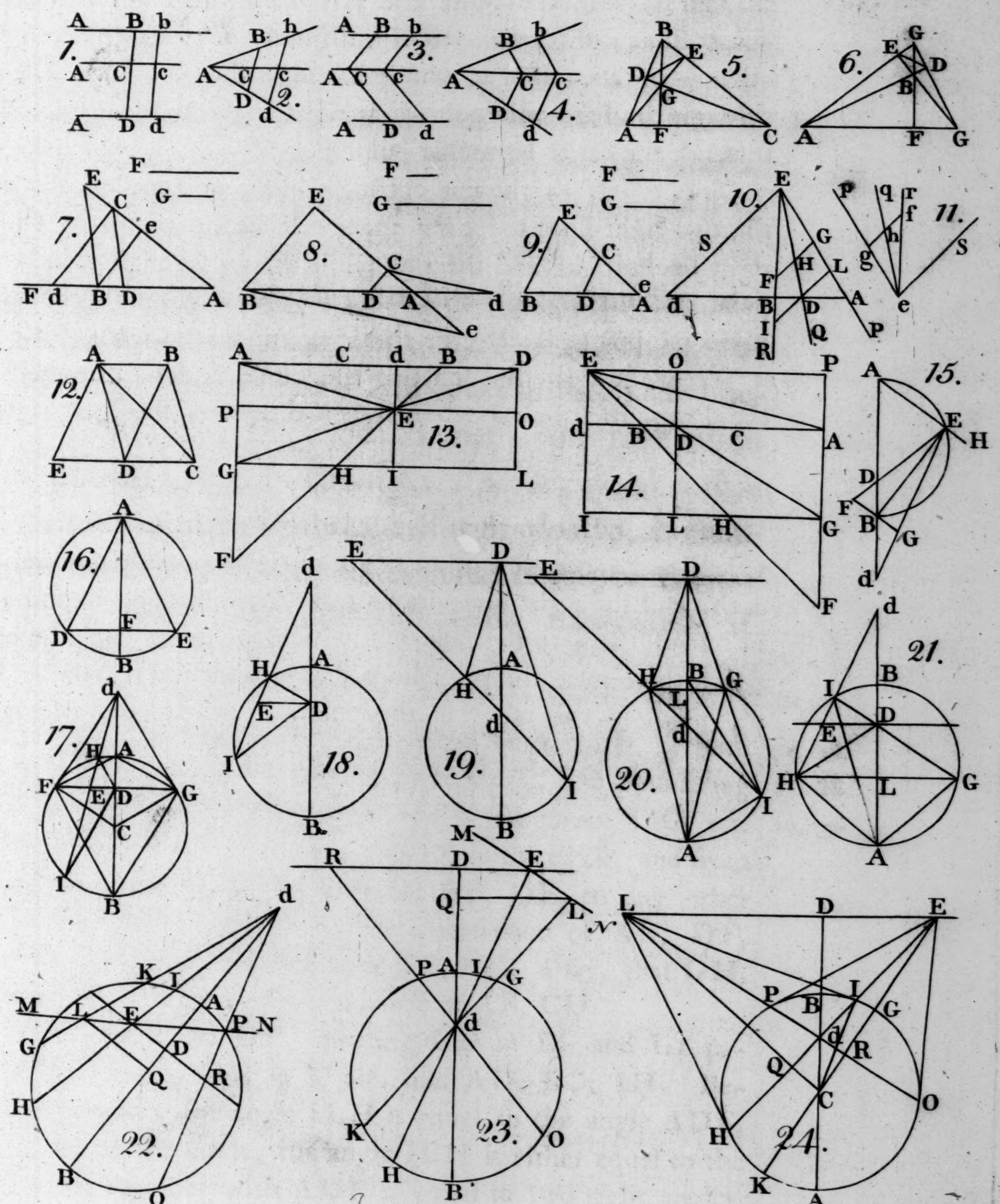
COR. 1. If a right line cutting a circle be harmonically divided, the two right lines touching the circle in the points in which the right line cuts it, and the right lines drawn through each harmonic point at right angles to the diameter drawn through the alternate harmonic point, shall verge to one common concourse.

When the right line cutting the circle is not a diameter, this is the Prop. itself; and when it is a diameter, the four right lines are each perpendicular to the diameters, and therefore being parallel between themselves, they verge to one common concourse.

COR. 2. If two right lines cutting a circle be harmonically divided, and meet each other in an inner harmonic point common to each, and the right lines drawn from the remaining harmonic point in one of them to the points in which the other cuts the circle do touch the circle, the right lines drawn from the remaining harmonic point in the other to the points in which the alternate harmonic line cuts the circle shall also touch the circle. 24.

Viz. The same things remaining, if LH, LI, touch the circle; I say, that PE, OE, shall also touch the circle. For because HI, PO, harmonically divided in D, E, and D, L, meet in the common harmonic point D, EL will be perpendicular to C D (1. COR. LEM. 10.) and because LH, LI, touch the circle, LC will be perpendicular to HI (LEM. 8.) Wherefore as in the Lem. CE is perpendicular to PO, and consequently the right lines touching the circle in P, O, shall verge to E, the concourse of LE, CE (1. COR.), viz. PE, OE, touch the circle.

L E M.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LEM. 12.

If two right lines inſcribed in a circle either touch or cut the circle, a point of contact being conſidered as a double point, and from two of their terms, one in each, be inflected two right lines to a point in the circle, and from the remaining terms be inflected two other right lines to any other point in the circle, the right line joining the interſections of the inflected right lines, viz. of one of each pair with the alternate one of the other pair, ſhall verge to the concourſe of the inſcribed right lines; that is, it ſhall be parallel to them, if they be parallel between themſelves, or it ſhall paſs through their concourſe, if they do meet.

FIG.

Let AB, CD, inſcribed in a circle, either both touch, or both cut, or the one touch and the other cut the circle, and conſidering a point of contact as a double point, if from the terms A, C, one in each, be inflected AE, CE, to a point E in the circle, and from the remaining terms B, D, be inflected BF, DF, to any other point F in the circle, and G be the interſection of AE, DF, and H of the remaining inflected lines CE, BF; I ſay, that GH, being joined, ſhall verge to the concourſe of AB, CD.

25, 26. 27, 28. 29, 30.

Draw HL parallel to DF, meeting CD in L, and LI parallel to AD, meeting AB in I, alſo join AD, BC, IH. Becauſe of the parallels, the angle ILH is equal to the angle ADF, and on account of the circle, the angle IBH is either equal to the angle ADF, or together with ADF is equal to two right angles.

Therefore

FIG. Therefore the angles ILH, IBH are either equal between themselves, or together are equal to two right angles, and consequently the four points I, L, B, H are in a circle (21. & 22. e. 3.). Again, on account of the parallels, the angle GLH is equal to the angle CDF, and the angle CBH is the same with the angle CBF; therefore the angles GLH, CBH, are together equal to the angles CDF, CBF. But on account of the circle the angles CDF, CBF, are together equal to two right angles, or are equal between themselves, and consequently the four points H, L, B, C, are also in a circle. Wherefore the five points I, L, B, C, H, are in the same circle, and the angles BIH and BCH or BCE are either equal between themselves, or are together equal to two right angles. But on account of the circle the angles BAE, BCE, are also equal between themselves, or together equal to two right angles, therefore the angle BIH is equal to the angle BAE, and consequently IH is parallel to AE (28. e. 1.). In each case
25, 26. therefore the triangles ILH, ADG are equiangular, and if AB be parallel to CD, because then AD is equal to IL, HL will be equal to GD, and being also parallel to GD, HG will be parallel to LD or CB (33. e. 1.), viz. it will verge to the concourse of
28, 29. AB, CD.—But if AB, CD meet, let them meet in R. Because
30. the triangles ADG, ILH are equiangular, GD is to HL (as AD is to IL, viz. on account of the parallels,) as RD is to RL, Wherefore RG, RH, are in one strait line (32. e. 6.), or GH verges to the concourse of AB, CD.

LEM. 13. PROBLEM.

A right line being given in position, and a point without it, through which a right line is drawn, it is required to find in this latter line a point, such that its perpendicular distance from the right line given in

in position may be to the distance of the same from the given point in a given ratio. FIG.

Let AB be a right line given in position, F a point given without it, and FE a right line passing through F, in which line FE suppose D to be the point required; viz. such that if DB be drawn perpendicular to AB, DB shall be to DF in a given ratio. 31, 32. 33, 34. 35, 36.

Draw FA perpendicular to AB, and let Z be a right line to which AF is in the given ratio. Because FA is given in magnitude, Z will be given in magnitude also. Now FE is either parallel to AB, or meets it. If it be parallel, DB will be equal to FA, wherefore because DB is to DF as FA is to Z, DF will be equal to Z, and consequently will be given in magnitude. But FE is given in position, and the point F is given, therefore the point D is given. 31.

If FE be not parallel to AB, let it meet it in E. Because DB is to DF as FA is to Z, and on account of the parallels, BD is to ED as FA is to EF; therefore ex æquo, ED is to DF as EF is to Z. But EF, AB, being given in position, and the point F being given, EF is given in magnitude (28. DATA.). Wherefore ED is to DF in a given proportion, and consequently EF being given in position and magnitude, the point D is given. 32, 33. 34, 35. 36.

LIMITS. If FE be parallel to AB, then since it is only required that FD be equal to Z, two points D, D, in the right line FE, whose distances from F are each equal to Z, will answer to the conditions of the problem. The problem therefore in this case always admits of two solutions, and the points D, D, are towards different parts of F, but towards the same parts of AB with F. 31.

If FE meet AB, let the limits be enquired into, when FA is equal to, greater, or less than Z.

First, if FE meet AB in A, then because ED or AD is to DF in the given ratio of EF or AF to Z, the problem will in this case be the same as in LEM. 3., and therefore accordingly as AF 37, 38. 39.

FIG. is greater or less than Z, there will be two points D, D, in AF, towards the same or different parts of A that F is, viz. on the same side of AB with F, or on different sides, which will answer the conditions of the problem; but if AF be equal to Z, only one point D, which is the bisection of AF, answers the conditions (1. COR. LEM. 3.).

32, 33. But if FE meet AB in any point E, other than A, then whether FA be equal to or greater than Z, yet in this case, because FE is greater than FA, it will be greater than Z, and therefore ED will be to DF in the ratio of a greater to a less, and the conclusion will be the same as in the last, viz. two points D, D, will answer to the conditions, and these will be in FE towards different parts of F, but towards the same parts of AB with F (1. COR. LEM. 3.).

When FA in this inclination of FE is less than Z, a circle described round F with a distance equal to Z, will meet AB in two points. Let this circle be described meeting AB in C, c, and

34. join FC, Fc. If FE fall without the angle CFc, EF will be greater than FC, viz. than Z, and therefore ED being to DF in the ratio of EF to Z, that is, of a greater to a less, two points D, D, will answer the conditions, and every conclusion will be the same as in the preceding.

35. If FE fall within the angle CFc, FE will be less than FC or Z, and ED being to DF in the ratio of FE to Z, or of a less to a greater, two points D, D, will answer the conditions of the problem, but these two points will now be in FE, towards different parts of E, and on the same side of F (1. COR. LEM. 3.). The problem therefore in this case also admits of two solutions, but the two points D, D, will be towards the same parts of F, and one will be on the part of AB opposite to F, the other on the same side with F.

36. Lastly, If FE coincide with FC or Fc, and therefore the point E with C or c, and FE become equal to FC or Z, then ED being to DF in the ratio of EF to Z, viz. of equality, there will be only one point D, which can answer to the conditions of

 the

the problem, and this point will be that wherein EF is bisected (1. COR. LEM. 3.). The problem therefore in this case admits only of one solution, and the single point D will be between F and B, and therefore on the same parts of AB with F. FIG.

COMPOSITION. Draw FA perpendicular to AB, and find the right line Z, such that FA may be to Z in the given ratio.

CASE 1. When FE is parallel to AB.—In FE take FD, F D, towards each part of F and each equal to Z. I say, that the points D, D, shall each answer to the conditions of the problem. Draw DB, D B, perpendicular to AB. DB is equal to AF, and DF to Z, therefore because equal magnitudes are proportional to equal magnitudes, DB is to DF as FA is to Z. And by the same reasoning is it shewn that D B is to D F as FA is to Z. 31.

CASE 2. When FE is not parallel to AB.—First, let it meet AB in A. In AF find a point D, such that AD may be to DF in the given ratio of AF to Z (LEM. 3.). If AF be either greater or less than Z, two points D, D, will be found to answer the conditions, and they will accordingly be on the same or different parts of A, viz. of AB, that the point A is. But if AF be equal to Z, only one point D, which is the bisection of AF, can answer the conditions (1. COR. LEM. 3.). 37, 38. 39.

Secondly, let AF meet AB in any point E other than A, and when FA is less than Z, round F with a distance equal to Z describe a circle meeting AB in C, c, and join FC, Fc. In EF find a point D, such that ED may be to DF in the given ratio of EF to Z (LEM. 3.). Also if AF be equal to or greater than Z, then EF being greater than AF will be greater than Z, and therefore two points D, D, both on the parts of E towards F, will be found to answer the condition (1. COR. LEM. 3.). But when AF is less than Z, if FE fall without the angle CFc, EF will be greater than FC, viz. than Z, and therefore the conclusion will 34, 35. 36. 32, 33. 34.

FIG. be the same as in the last instance. If FE fall within the angle
35. CFc, then EF will be less than FC, viz. than Z, and therefore two points D, d, will also answer the condition, but these two points will be in EF towards different parts of E (1. Cor.
36. Lem. 3.). Lastly, if FE coincide with FC, or Fc, viz. if the point E fall in C or c, and EF become equal to FC or Z, then the ratio of ED to DF being that of equality, only one point D, which is the bisection of EF, will answer the condition.

In each case, on account of the parallels, ED is to DB as EF is to FA, and also by construction, ED is to DF as EF is to Z. Therefore, ex æquo, DB is to DF as FA is to Z, viz. in the given ratio. And by the same reasoning it is shewn that db is to dF in the given ratio.

31. Cor. 1. If FE be parallel to AB, two points D, d, towards opposite parts of F, will universally be found to answer the conditions of the problem, whether the given ratio be that of a greater to a less, of a less to a greater, or of equality.

33. Cor. 2. If FE meet AB, and the given ratio be that of a
34. greater to a less, or, being that of a less to a greater, if it fall without the angle CFc, or being that of equality, if it meet AB
32. in any other inclination than of a right angle two points will also answer to the conditions, and these two points will be on the same side of AB with F, but will be in FE towards different parts of F.

35. Cor. 3. If FE meet AB, and fall within the angle CFc, when the given ratio is that of a less to a greater, there will still be two points which answer to the conditions, but these two points will in this case be on different sides of AB, and in FE will be towards the same parts of F.

36. Cor. 4. If FE coincide with FC or Fc, when the given ratio
39. is that of a less to a greater, or meet AB at right angles when the given ratio is that of equality, only one point will answer the conditions, and that point will be the bisection of FE or FA.

LEM.

LEM. 14.

If there be two circles, whose ſemidiameters are proportional to the perpendicular diſtances of their centres from a right line given in poſition, and from the centres any two parallels be drawn to meet the right line given in poſition, the rectangles under the ſegments of the parallels, intercepted between the right line and the circumference of each circle, ſhall be to each other as the ſquares of the ſemidiameters.

FIG. 40, 41. 42.

Let ADE, ADE, be two circles, whoſe centres are S, s, and PR a right line, on which are drawn the perpendiculars SR, SR, and alſo from S, s, are drawn the parallels SP, SP, meeting RP, RP, in P, P, and the circumferences in D, E, and D, E. Then if SD be to SR as SD is to SR; I ſay, that the rectangle DPE ſhall be to the rectangle DPE as the ſquare of SD is to the ſquare of SD.

On account of the ſimilar triangles SPR, SPR, SP is to SP as SR is to SR, and, by hypotheſis, SD is to SD as SR is to SR; therefore ex æquo, SP is to SP as SD is to SD. Whereſore SD, SD, are proportionally cut in P, P, and becauſe DE, DE, are biſected in S, s, they are alſo proportionally cut in E, E. PD is therefore to PD as SD is to SD, and PE is to PE as SD is to SD, and compounding theſe ratios, the rectangle DPE is to the rectangle DPE as the ſquare of SD is to the ſquare of SD.

COR. 1. The rectangles DPE, DPE, are alſo as the ſquares SP, SP, becauſe SP, SP, are proportional to SD, SD.

COR. 2. The rectangles under the ſegments of any two right lines drawn from the points P, P, to cut the circles, or the ſquares of

FIG. of the right lines drawn from P, p, to touch the circles, are as the squares of SP, sp, or SD, sd.

For these rectangles or these squares are equal to the rectangles DPE, dpe.

LEM. 15.

43. If from a point I without a circle ADB, whose centre is C, be drawn the diameter IAB, and also ID to meet the circumference in D, and from D be drawn DF perpendicular to AB, the rectangle AIB together with the square of DI shall be equal to twice the rectangle FIC.

Draw CE perpendicular to ID, and in IE produced, towards the parts of E opposite to I make Ei equal to EI. Then iI is double to EI, and the rectangle iID is equal to twice the rectangle EID. But the rectangle iID is equal to the rectangle iDI together with the square of DI (3. e. 2.); therefore the rectangle iDI together with the square of DI is equal to twice the rectangle EID. Let ID meet the circle again in d. Because Ei is equal to EI, and ED to Ed (3. e. 3.), the remainder or the whole iD shall be equal to the remainder or the whole Id, and the rectangle iDI be equal to the rectangle dID. Therefore the rectangle dID together with the square of DI is equal to twice the rectangle EID. But because the angles at E, F, are right, the four points C, E, D, F, are in a circle, and the rectangle FIC is equal to the rectangle EID; also the rectangle AIB is equal to the rectangle dID (Cor. 36. e. 3.). Wherefore the rectangle AIB together with the square of DI is equal to twice the rectangle FIC.

Cor. If ID touch the circle, in which case the points D, E, d, fall into one, the demonstration is the same, and the conclusion the same.

LEM.

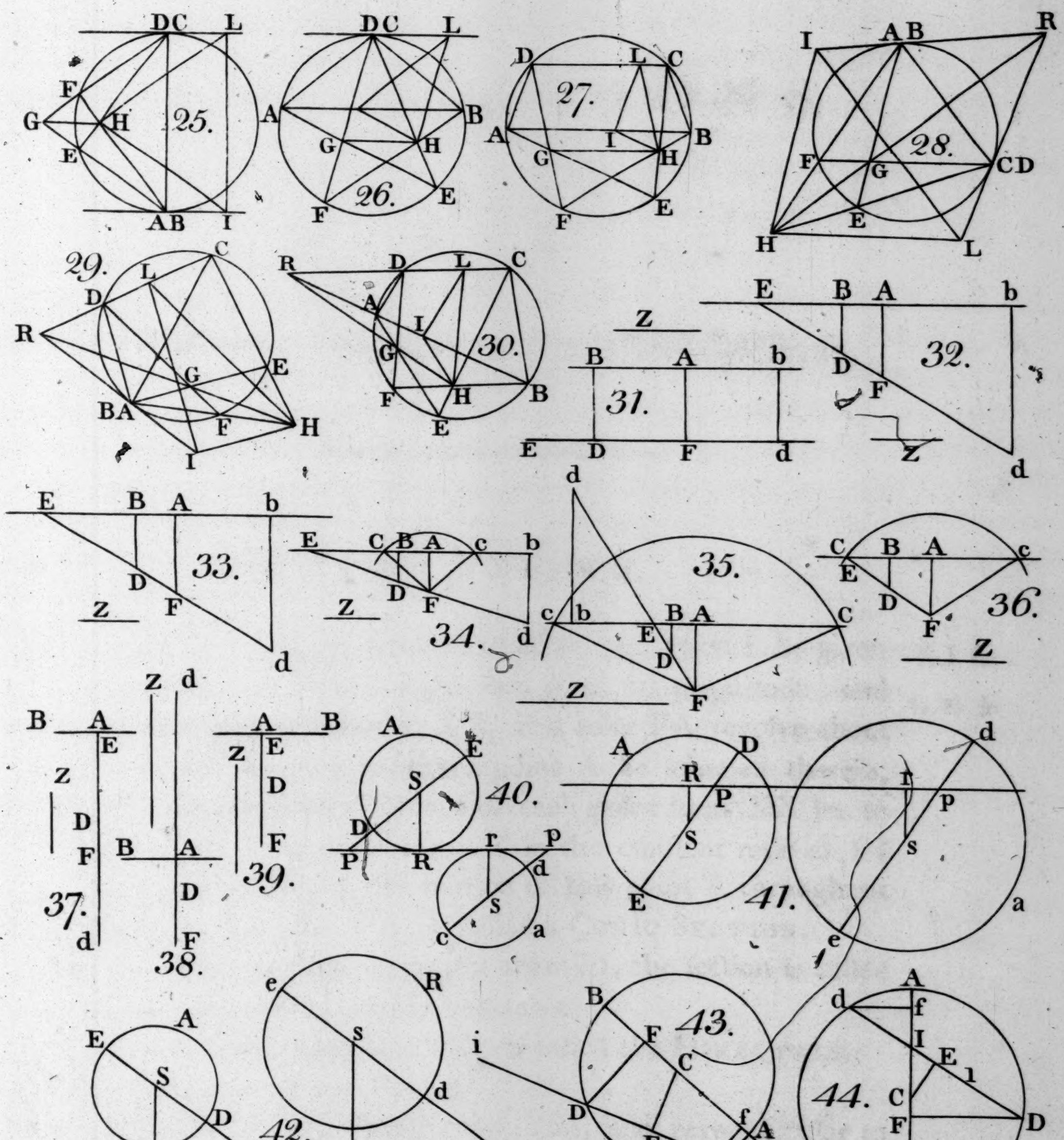
LEM. 16.

FIG.

If through a point I within a circle a right line DId be drawn meeting the circumference in D, d, and other things remain the same as in the preceding, the square of the greater segment DI shall exceed the rectangle AIB by twice the rectangle FIC; but the square of the less segment dI shall be less than the rectangle AIB by twice the rectangle fIC. 44.

The square of DI is equal to the rectangle iDI together with the rectangle iID (2. e. 2.). But for the same reason as in the preceding, the rectangle iDI is equal to the rectangle DId, viz. to the rectangle AIB, and the rectangle iID is equal to twice the rectangle EID, viz. to twice the rectangle FIC. Therefore the square of DI is equal to the rectangle AIB together with twice the rectangle FIC.

But in the case of the segment dI, the square of dI together with the rectangle iId is equal to the rectangle idI (3. e. 2.), while for the same reason as above the rectangle idI is equal to the rectangle DId, viz. to the rectangle AIB, and the rectangle iId is equal to twice the rectangle EId, viz. to twice the rectangle fIC. Therefore the square of di together with twice the rectangle fIC is equal to the rectangle AIB. 44.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CONIC SECTIONS.

LIB. I.

Of the Properties common to the Three Sections.

DEFINITIONS.

1. IF a right line XX be given in position, the point F be given without it, and Z be a right line given in magnitude; and FI being drawn perpendicular to XX, if a ruler FA revolve about the point F, and in every position a point A be assumed therein, such that the perpendicular distance of each point from XX be to the distance of the same point from F in the constant ratio of FI to Z; the line described by the motion of this point A throughout the revolution of the ruler FA, is called a CONIC SECTION. FIG. 1, 2, 3.

2. As Z is equal to, less, or greater than FI, the section is called a PARABOLA, ELLIPSE, or HYPERBOLA.

3. The immoveable right line XX is called the DIRECTRIX.

4. The fixed point F the FOCUS.

5. A right line passing through the focus, and perpendicular to XX, viz. whose position is the same with FI, is called the AXIS of the parabola, the TRANSVERSE AXIS of the ellipse and hyper-

FIG. bola, the MAJOR AXIS of the ellipse, or without distinction the FOCAL AXIS of each section.

6. The right line Z is called the SEMI-LATUS RECTUM, or SEMI-PARAMETER of the focal axis, or the PRINCIPAL SEMI-PARAMETER.

7. A point is said to be IN the section, through which the section passes; WITHIN the section, when every right line drawn through it meets the section; but WITHOUT the section, when a right line may be drawn through it, which however extended, does not meet the section.

8. A right line is said to TOUCH a conic section which meets it, but however produced towards both parts of the concourse, falls in every other point without the section.

9. Two conic sections, or any two curve lines, are said to TOUCH each other, when the same right line touches them both in the common point in which they meet each other.

NOTE. The sense of the terms in, within, without, in the 7. DEF. is applied to the circle. Thus if a point be said to be in a circle, it is understood that it is in the circumference. Euclid by the term circle means the space comprehended by the circumference; but geometric language would have been simpler, if in the same manner as in the ellipse, the term of circle had been applied only to the line which bounds the space. Pure geometry converses little, if at all, with the spaces comprehended by curve lines.

PROP. I.

If from any point in a conic section and from the focus two parallel right lines be drawn to meet the directrix in any angles, the parallel drawn from the point in the section shall be to the focal distance of the

the point, as the parallel drawn from the focus is to the semi-latus rectum of the axis. FIG.

And conversely, if there be a point in the plane of a conic section, from which and from the focus are drawn two parallels to meet the directrix in any angles; then, if the parallel from the point be to the focal distance of the point, as the parallel from the focus is to the semi-parameter of the axis, the point shall be in the section.

Let A be any point in a conic section, whose focus is F, directrix XX, and principal semi-parameter Z. Then first, let the parallels drawn from A, F, viz. AD, FI, be perpendicular to XX, and AF be joined. I say, that AD shall be to AF as FI is to Z. For AF is one position of the revolving ruler, and therefore, A being a point in the section, is one of the points assumed in the I. DEF., and consequently, AD is to AF as FI is to Z. 1, 2, 3.

Now let the right lines AP, FR, parallel to each other, meet the directrix in any angles; I say that AP shall be to AF as FR is to Z.

For, the same things remaining, because of the parallels, AD is to AP as FI is to FR; and it has been shewn that AD is to AF as FI is to Z; therefore ex æquo, AP shall be to AF as FR is to Z.

Conversely, if from a point A in the plane of the section be drawn AD perpendicular to the directrix, or AP any how to meet it, and FI, FR, accordingly parallel to AD, AP, and AF be joined; I say, that if AD be to AF as FI is to Z, or if AP be to AF as FR is to Z, the point A shall be in the section.

First, because AF is one position of the describing ruler, and AD is to AF as FI is to Z, therefore A is one of the points through which, according to the I. DEF. the section passes; viz. the point A is the section

FIG. Again, if AP be to AF as FR is to Z, the point A shall be in the section.—For, the same things remaining, because of the parallels, AP is also to AD as FR is to FI; wherefore, ex æquo, AD is to AF as FI is to Z, and consequently the point A is in the section.

COR. 1. If from two points in the plane of a conic section be drawn two parallels to meet the directrix, and the parallel from each point be to the focal distance of each in the same ratio; then if one of the points be in the section, the other shall be in the section also.

Conversely, if each point be in the section, the parallels from the points shall be to the focal distances of the points in the same ratio.

2. Let KL, AP, parallel to each other, meet the directrix in L, P, and KF, AF be joined. If KL be to KF as AP is to AF, and the point K be in the section, the point A shall be in the section also. Draw FR parallel to KL or AP, meeting the directrix in R. LK is to KF as FR is to Z; therefore ex æquo, AP is to AF as FR is to Z, and consequently the point A is in the section.

Conversely, if both the points K, A, be in the section, KL shall be to KF as AP is to AF. For AP is to AF as FR is to Z, and KL is to KF as FR is to Z; therefore, ex æquo, KL is to KF as AP is to AF.

COR. 2. If a right line, meeting a conic section in two points, meet the directrix, the segment intercepted between each point and the directrix shall be to the focal distance of each point in one and the same ratio.—And conversely, if the segment intercepted between each point and the directrix be to the focal distance of each point in one and the same ratio, and one of the points be in the section, the other point shall be in the section also.

This is only a particular case of the preceding corollary.

COR. 3. If a right line drawn from the focus to the directrix, meet a conic section in two points, it shall be harmonically divided in its concourses with the section.

Conversely,

Converſely, if the right line be harmonically divided, and one of the harmonic points be in the ſection, the other harmonic point ſhall be in the ſection alſo. FIG.

Thus if FR drawn from the point F meet the directrix in R, and the ſection in E, G; then as a caſe of the 1. COR. GR is to FG as ER is to EF, viz. FR is harmonically divided in E, G. 1, 2, 3.

And converſely, if FR be harmonically divided in E, G, and one of theſe points, as E, be in the ſection, the other point G ſhall be in the ſection alſo.

For becauſe of the harmonic diviſion, GR is to FG as ER is to EF, therefore the point E being in the ſection, the point G ſhall be in it alſo (2. COR.)

COR. 4. If a right line parallel to the directrix meet a conic ſection in two points, the diſtances of the ſaid points from the focus ſhall be equal between themſelves.

Converſely, if the focal diſtances of two points in a right line parallel to the directrix be equal between themſelves, and one of the points be in the ſection, the other ſhall be in it alſo.

Let AA parallel to the directrix meet a conic ſection in A, A, and to the focus F be drawn AF, AF; I ſay that AF ſhall be equal to AF. 6.

Draw AB, AB, perpendicular to the directrix. AB is to AF as AB is to AF (1. COR.); but on account of the parallels, AB is equal to AB, therefore AF is equal to AF.

And converſely, AA being parallel to the directrix, if AF be equal to AF, and the point A be in the ſection, the point A ſhall be in it alſo.—For AA being parallel to the directrix, AB is equal to AB, and AB is to AF as AB is to AF. Therefore the point A ſhall be in the ſection (1. COR.)

COR. 5. If through the focus of a conic ſection be drawn a parallel to the directrix, either ſegment intercepted between the focus and the ſection ſhall be equal to the ſemi-parameter of the focal axis, and therefore the whole parallel be equal to the whole parameter.

Through

FIG. 6. Through the focus F draw HFh parallel to the directrix, meeting the section in H, h, and draw HL, hl, FI, perpendicular to the directrix, also let Z be the semi-parameter of the focal axis. HL is to HF as FI is to Z. But, because of the parallels, HL is equal to FI, therefore HF is equal to Z. For the same reason is hF equal to Z, therefore the whole HFh is double to Z, viz. is equal to the whole parameter.

COR. 6. In the parabola, the perpendicular distance of any point in the section is equal to the focal distance of the point.

For the perpendicular distance is to the focal distance in the ratio of FI to Z, viz. in the ratio of equality (2. DEF.).

COR. 7. In the parabola, the portion of the axis intercepted between the focus and directrix is bisected in the vertex, and each segment is equal to a fourth part of the latus rectum.

Let the axis of the parabola meet the section in E, the directrix in I; I say, that FI shall be bisected in E, and FE, EI shall each be equal to a fourth part of the latus rectum, or to the half of Z. For FE is equal to EI (6. COR.), and the whole FI is equal to Z; therefore FE, EI, are each equal to the half of Z, viz. to a fourth part of the latus rectum.

PROP. 2.

The parabola and ellipse are situated wholly towards the same parts of the directrix with the focus; but the hyperbola consists of two parts, one towards the same parts of the directrix with the focus, the other towards the opposite; and these two parts of the hyperbola are called the OPPOSITE HYPERBOLAS or SECTIONS.

In

In the parabola and ellipſe, becauſe the given ratio is either that FIG.
of equality or of a greater to a leſs, the points in the deſcribing ruler, whoſe perpendicular diſtances from the directrix are to their focal diſtances in the given ratio, will always be towards the ſame parts of the directrix with the focus (2. COR. LEM. 13.). Therefore the parabola and ellipſe are ſituated wholly towards the ſame parts of the directrix with the focus.

But in the hyperbola, becauſe the given ratio is that of a leſs to 5.
a greater (2. DEF.), viz. becauſe Z is greater than FI, therefore a circle deſcribed round the centre F with the diſtance Z will cut the directrix in two points K, k. Then in every poſition in which the deſcribing focal ruler falls within the angle KFk, there will, as in the ellipſe, be two points therein, whoſe perpendicular diſtances from the directrix are to their focal diſtances as FI to Z, but only one of theſe points will be towards the ſame parts of the directtrix with the focus, while the other will be towards the oppoſite parts (3. COR. LEM. 13.). Wherefore while the deſcribing ruler moves within the angle KFk, one of the points is deſcribing an hyperbola on the oppoſite ſide of the directrix, and the other an hyperbola on the ſame ſide with the focus.

When the deſcribing ruler paſſes into the ſituation of FK or Fk, the point towards the oppoſite parts of the directrix vaniſhes, and the point on the ſame ſide of the directrix with the focus, and which then alone anſwers to the conditions, falls in the biſection of FK or Fk (4. COR. LEM. 13.). In this moment therefore the ſection on the oppoſite ſide of the directrix vaniſhes, but the other proceeds and paſſes through the biſection of FK or Fk.

When the deſcribing ruler proceeding in its motion falls without the angle KFk, then two points again commence, which anſwer the conditions, but theſe two points are now both towards the ſame parts of the directrix with the focus (2. COR. LEM. 13.), viz. they are in the ſection ſituated on the ſame ſide with the focus. Wherefore the deſcribing ruler in the whole of its motion without

FIG. the angle KF*k* continues the description of the hyperbola, that is situated towards the same parts of the directrix with the focus.

COR. 1. The axis of the parabola meets the section in one point only, and the portion thereof intercepted between the focus and directrix is bisected in that point, viz. in the vertex (4. COR. LEM. 13.). This follows also from the equality of the given ratio, as appeared in the 7. COR. 1.

COR. 2. In the hyperbola, every right line drawn through the focus, and falling within the angle KF*k*, meets the opposite sections, each in one point; the one, between the focus and directrix, the other, in a point towards the opposite parts of the directrix. But the right lines FK, F*k*, meet each that section alone, which is situated on the same side of the directrix with the focus, and each in one point only, and are each bisected in that point.

COR. 3. In the parabola, every right line drawn through the focus, except the axis; in the hyperbola, every right line falling without the angle KF*k*; in the ellipse every right line whatever passing through the focus; meets the same section in two points, towards different parts of the focus. The parabola and hyperbola therefore are concave towards the focus, but have no existence towards one part of it, while the ellipse is concave towards the focus on all sides, surrounds it, and encloses a space. Vide LEM. 13.

COR. 4. The right line FK or F*k*, the semidiameter of the circle described round the focus of the hyperbola, is equal to the principal semi-parameter.

DEFINITIONS.

10. In the ellipse and hyperbola, if the portion of the focal axis, intercepted by the section or sections be bisected; the point of bisection is called the CENTRE.

11. Every

FIG.

11. Every right line drawn through the centre of the ellipse and hyperbola, and every right line perpendicular to the directrix of the parabola, is called a DIAMETER of the section.

12. The diameter of the ellipse and hyperbola parallel to the directrix, or perpendicular to the focal axis, is called the AXIS SECUNDUS, or the MINOR AXIS of the ellipse.

13. The point in which a diameter meets a conic section, is called a VERTEX of that diameter.

14. Two right lines CM, Cm, drawn through the centre of an hyperbola, parallel to FK, Fk, are called the ASSYMPTOTES of the hyperbolas. 35.

15. Every diameter of the hyperbolas, falling within those two vertical angles of the assymptotes, which comprehend the opposite hyperbolas, is called a TRANSVERSE DIAMETER; but a diameter falling within the adjacent vertical angles, is called a DIAMETER SECUNDA.

NOTE. This and the 12. DEF. regard only the position of the diameters, but if the magnitude of a diameter be considered, then by every diametet of the ellipse, and every transverse diameter of the hyperbola, is understood that portion which is intercepted by the section or sections.

But in the case of a diameter secunda of the hyperbola, as DCd, 5.
suppose a right line EG, parallel to Dd, to touch either section in E, and meet an assymptote in G; make CD, Cd, towards each part of the centre C, equal to EG; then by the magnitude of the diameter secunda DCd, is understood the portion Dd.

16. A circle described round any point in the plane of a conic section, whose semidiameter is to the distance of its centre from the directrix as the semi-parameter of the focal axis is to the distance of the focus from the directrix, is called a GENERATING CIRCLE.

NOTE. It is the same, whether the distances be measured perpendicularly, or by any two parallels, drawn from the centre and focus to the directrix.

FIG.

17. Two right lines, the one drawn from the focus, the other from any point in the plane of a conic section, and either both meeting the directrix in the same point, or both parallel to the directrix, are called RESPONDENT LINES, or simply RESPONDENTS; and the one drawn from the focus is called the FOCAL RESPONDENT, the other the CONIC RESPONDENT.

18. Two right lines parallel to two respondents, and either both meeting the directrix in the same point, if the respondents meet the directrix, or parallel to the directrix, if the respondents be parallel thereto, and in that case having their distances from the directrix proportional to the distances of the respondents from the directrix, whether these distances be measured perpendicularly or by any two parallel lines, are called CORRESPONDENTS; and they are distinguished as FOCAL or CONIC CORRESPONDENT, according to the respondent to which each is parallel.

19. If there be two respondents, and two parallels be drawn, one from the focus to meet the conic respondent, the other from any point in the conic correspondent to meet the focal respondent, the two points of concourse are called RESPONDENT POINTS.

20. If there be two respondents, and any two right lines correspondent to them, and two parallels be drawn, one from the focus to meet the conic respondent, the other from any point in the conic correspondent to meet the focal correspondent, the two points of concourse are called CORRESPONDENT POINTS.

21. If round any point in a conic respondent or conic correspondent the generating circle be described, the two respondents or two correspondents are said to pertain to the generating circle, or to be respondents or correspondents in the circle.

COROLLARIES from the DEFINITIONS.

1. If from the centre of a generating circle and from any point in a conic section two right lines parallel to each other be drawn to

meet the directrix, the semidiameter of the circle shall be to the parallel drawn from the centre as the focal distance of the point is to the parallel drawn from the same point.—For, drawing a parallel also from the focus to meet the directrix, the semidiameter is to the parallel drawn from the centre as the principal semi-parameter is to the focal parallel (16. DEF.), and the focal distance of the point is to the parallel drawn from the point in the same ratio (1.); therefore the COR. immediately follows. FIG.

2. If a right line drawn through the focus, and meeting the section or opposite sections in two points, be bisected; the generating circle described round the point of bisection shall pass through the two points in which the right line meets the section or sections; that is, the semidiameter of the generating circle shall be equal to half the focal line.

Let GE drawn through the focus F of a conic section meet the section in G, E, and be bisected in O. I say, that the generating circle described round O passes through G, E. First, let GE meet the directrix in R. Because GE is harmonically divided in F, R (3. COR. 1.), and bisected in O, OE will be to OR as FE is to ER (1. COR. LEM. 5.), and therefore OE or OG is equal to the semidiameter of the generating circle described round O (1. COR.). 1, 2, 3.

If Hh parallel to the directrix pass through the focus F, and meet the section in H, h. Then Hh is bisected in F, and HF or hF is equal to the principal semi-latus rectum (5. COR. 1.). Therefore HF or hF is to FI as the principal semi-parameter is to FI, and consequently HF or hF is equal to the semidiameter of the generating circle described round F. 6.

3. Hence it appears that the semidiameter of the generating circle pertaining to the focus, is equal to the principal semi-latus rectum.

4. Hence also it appears, that the semidiameter of the generating circle pertaining to the centre of the ellipse and hyperbola is equal to the semi-axis transverse.

FIG. 5. The generating circle pertaining to a point in a conic section passes through the focus, viz. the semidiameter is equal to the focal distance of the point.

1, 2, 3. From a point A in a conic section draw AF to the focus, and AD perpendicular to the directrix. Then AF is to AD as Z is to FI (1.), therefore AF is equal to the semidiameter of the generating circle described round A (16. Def.), viz. the circle passes through F.

6. The directrix touches every generating circle pertaining to the parabola.

For the principal semi-parameter being in the case of the parabola equal to the perpendicular distance of the focus from the directrix (2. Def.), the semidiameter of every generating circle is equal to the perpendicular distance of the centre from the directrix (16. Def.), and therefore the generating circle described round any point pertaining to the parabola, touches the directrix.

7. Every generating circle of the ellipse falls entirely without the directrix, but of the hyperbola meets the directrix in two points.

Because in the ellipse Z is less than FI (2. Def.), therefore the semidiamer of every generating circle pertaining to this section is less than the perpendicular distance of its centre from the directrix (16. Def.), and consequently, the circle falls entirely without the directrix.—But in the hyperbola, Z is greater than FI, and therefore the semidiameter being greater than the perpendicular distance of the centre from the directrix (16. Def.), the circle cuts the directrix in two points.

8. If from any point in the plane of an hyperbola a right line parallel to either assymptote be drawn, and meet the directrix, the right line so drawn shall be a semidiameter of the generating circle pertaining to that point; and therefore right lines drawn from the centre of any generating circle of the hyperbola to the two points in which the circle meets the directrix are parallel to the assymptotes,

totes, and consequently parallel to the right lines drawn in like manner from the centre of any other generating circle. FIG.

The same things remaining as in PROP. 2., from any point S draw SO, So, parallel to the assymptotes CM, Cm, and meeting the directrix in O, o. The semidiameter of the generating circle round S is to SO or So, as Z is to FK or Fk (16. DEF.). But FK or Fk is equal to Z (4. COR. 2.). Therefore SO, or So, is the semidiameter of the generating circle described round S, and therefore SO, So, drawn from the centre S of a generating circle to the points O, o, in which the circle meets the directrix are parallel to the assymptotes. In like manner will it appear, that the right lines AP, Ap, drawn from the centre A of any other generating circle are parallel to the assymptotes, and therefore parallel to SO, So. 7.

9. Hence the generating circle pertaining to the centre of an hyperbola passes through the concourses of the assymptotes with the directrix, and therefore the segment of each assymptote intercepted between the centre and the directrix, is equal to the semiaxis transverse. 5.

By the preceding COR. the generating circle described round the centre C passes through M, m, and by the 4. COR. it passes also through E, H; therefore CM or Cm is equal to CE or CH.

10. If a correspondent to one of two respondents be drawn, only one right line can be correspondent to the other respondent.

For the correspondents are either drawn through one and the same point in the directrix parallel to the same right lines; or if from the parallelism of the respondents to the directrix, they must themselves also be parallel to the directrix, yet their distances from the directrix are proportional to the distances of the respondents. Therefore the respondents and one correspondent being given, only one right line can be the other correspondent.

11. If two respondents and their correspondents meet the directrix, any two parallel right lines falling upon each pair and the directrix shall be divided in the same ratio; but if they be all parallel

FIG. rallel to the directrix, then any two right lines, whether parallel or not, falling upon them and the directrix, shall be divided in the same ratio.

9. Let AP, FP respondent to each other meet the directrix in P, and Sp, Ap, correspondent to them meet the directrix in p. I say, that any two parallels GH, gh, meeting them and the directrix in G, A, H, and g, a, h, shall be divided in the same ratio. For, because of the parallels, AH is to Sh (as AP is to Sp, viz.) as AG is to Sg.

8. But, if AP, FP, and their correspondents Sp, Ap, be parallel to the directrix, then whether GH, gh be parallel or not, they shall be divided in the same ratio.

For AH, GH, are in the proportion of the perpendiculars drawn from A, G, to the directrix, and Sh, gh, are in the proportion of the perpendiculars drawn from S, g, to the directrix. But these perpendiculars are in the same ratio (18. Def.), therefore AH is to GH as Sh is to gh.

12. If the focal correspondent pertaining to a generating circle be given, the right line drawn from the centre of the circle to meet it in the directrix, or parallel to the directrix, accordingly as itself meets or is parallel to the directrix, shall be the conic correspondent.

13. If two conic respondents be parallel between themselves, but not parallel to the directrix, any two focal correspondents to them, pertaining to a generating circle, shall meet in the directrix. And conversely.

20, xii, 1. Let AP, BR, the conic respondents, whose focal respondents are FP, FR, be parallel to each other, and meet the directrix in P, R; then, in any generating circle whose centre is S, the right line Sp parallel to AP or BR, will be the conic correspondent to each of them. Therefore pa, pb, drawn from the same point p in the directrix and parallel to FP, FR, will be the focal correspondents to AP, BR.

Conversely,

Conversely, If two focal correspondents PA, PB, pertaining to the circle meet the directrix in the same point P, their respondents AP, BR shall be parallel between themselves. FIG.

For, every thing remaining the same, join SP, FP, FR. Then it appears, that AP, BR, are parallel to the same right line SP, and therefore are parallel between themselves.

14. If two conic respondents meet each other in the directrix, their focal correspondents pertaining to a generating circle shall be parallel between themselves.

Let AP, BP, which are conic respondents to the common focal respondent FP, meet the directrix and each other in the point P; I say, that their focal correspondents pertaining to a generating circle, shall be parallel between themselves. 20, 11, 2.

From S the centre of any such circle draw SP, SR, parallel to AP, AR, meeting the directrix in P, R. Then PA, RB, the focal correspondents to AP, BR, must be parallel to the same right line FP, and therefore parallel between themselves.

15. The focus, directrix, and principal semi-parameter of a conic section, or the focus, directrix and any point in the section, being given; the generating circle round any given point may be described.

Let F be the focus, XX the directrix, Z the principal semi-parameter, A any point in a conic section, and S be a point round which it is required to describe the generating circle. Draw in any direction the parallels FK, AP, SO, meeting the directrix, and join AF. Assume a fourth proportional to FK, Z, SO, or to AP, AF, SO; the circle described round S with the distance of this fourth proportional shall be the generating circle required (16. DEF. & 1. COR.). 7.

16. Two respondents being given, their correspondents pertaining to a given generating circle may be drawn. And conversely.

If the given respondents AP, FP, meet the directrix in P, from S the centre of the given generating circle draw SP parallel to the conic respondent AP, meeting the directrix in P, and draw 9.

PA

PA parallel to FP. Sp, ap, will be correspondent to AP, FP (18. Def.)

8. But, if AP, FP, be parallel to the directrix, draw AH meeting AP in A, FP in G, and the directrix in H; also draw Sh to the directrix. In Sh find the point g, such that AH may be to GH as Sh is to gh (10. e. 6.), and draw gp parallel to the directrix. Then Sp, gp, shall be the correspondents to AP, FP, pertaining to the circle (18. Def. & 10. Cor.).—The Converse is obvious.

17. The semiaxis transverse of the ellipse and hyperbola is a mean proportional between the distances of the centre from the directrix and focus. Also, the rectangle under the segments of the axis between the vertices and directrix is equal to the rectangle under the segments between the centre and directrix, the directrix and focus. And, the rectangle under the segments between the vertices and focus, is equal to the rectangle under the segments between the centre and focus, the focus and directrix.

18. If the generating circle pertaining to the centre of the ellipse and hyperbola be described, and through the focus of the ellipse be drawn a parallel to the directrix, meeting the circle in two points; then, in the ellipse, right lines drawn from the concourse of the axis with the directrix to these two points, shall touch the circle; but, in the hyperbola, right lines drawn from the focus to the two points in which the circle cuts the directrix, shall touch the circle.

19. Hence, in the hyperbola, the right lines drawn from the focus to the concourse of the assymptotes with the directrix, will be perpendicular to the assymptotes, and will each be equal to the semi-axis secundum.

4, 5. Every thing remaining as in Prop. 2.; these three last corollaries follow from the harmonic division of HE in the points I, F (3. Cor. 1.), and from the bisection of the axis HE in the centre C.

Thus if, in the ellipse, MFm be drawn parallel to the directrix, meeting the circle in M, m; and IM, Im be joined; but, in the hyperbola, if the generating circle meet the directrix in M, m, and FM, Fm, be joined;

Then

FIG.

Then in each ſection, becauſe of the harmonic diviſion, and biſection of HE in C, CE or CH is a mean proportional between CI, CF; the rectangle HIE is equal to the rectangle CIF, and the rectangle HFE is equal to the rectangle CFI.—This is the 17. COR.

Again, becauſe Mm is drawn through one of the harmonic points at right angles to the diameter of the circle; therefore IM, Im, in the ellipſe, and FM, Fm, in the hyperbola, touch the circle.—This is the 18. COR.

Laſtly, becauſe in the hyperbola, FM, Fm, touch the circle, they are perpendicular to the ſemidiameters CM, Cm (COR. 16. e. 3.), and are alſo each equal to the ſemiaxis ſecundus. Draw EG parallel to Dd the axis ſecundus, viz. perpendicular to EH, meeting either aſſymptote in G. Becauſe the triangles CMF, CEG, have a common angle at C, as alſo each a right angle at M, E, and the homologous ſides CM, CE are equal between themſelves, two other homologous ſides FM, EG will alſo be equal between themſelves (26. e. 1.). But EG is equal to the ſemiaxis ſecundus (NOTE to 15. DEF.) *, therefore FM is alſo equal to the ſemiaxis ſecundus.—This is the 19. COR.

PROP. 3.

If a right line meet a conic ſection, and round any point therein, or in a conic correſpondent thereto a generating circle be deſcribed, the focal reſpondent or focal correſpondent ſhall meet the circle in a reſpondent point. And Converſely.

* NOTE. It will afterwards be proved that the right line parallel to Dd and touching either hyperbola, paſſes through E or H.

FIG. 1. When the centre of the generating circle is in the right line itself.

10, 11. Let the right line AP meet a conic section in the point A, and round any point S in AP a generating circle be described; I say, that FP the focal respondent to AP shall meet the circle in a point respondent to A.

Let F be the focus, XX the directrix, and Z the principal semiparameter of the section. Join FA, and parallel thereto draw S A, meeting FP in A. Then A is a point respondent to A (19. DEF.), but I say also, that the point A is in the circle. To the directrix draw the parallels AD, SE, FI. Because AP, FP, XX, either meet in one common point P, or are mutually parallel to each other, and from two points S, A, in AP are drawn the parallels S A, AF, to FP, and the parallels SE, AD, to XX; therefore S A will be to SE as AF is to AD (LEM. 1.), and consequently S A is equal to the semidiameter of the generating circle described round S (1. COR. DEF.). Wherefore the point A is in the circle, viz. the focal respondent FP does meet the circle in a point A respondent to A.

CASE 2. When the centre of the generating circle is in a conic correspondent to the right line.

12, 13. The same things remaining, let S P, A P, be any two correspondents to AP, FP, of which latter the conic respondent AP meets
14. the section in A, and round any point S in S P, the conic correspondent, let a generating circle be described. I say, that P A will meet this circle in a point respondent to A.—Join AS, and draw FR respondent thereto, meeting P A in A, and join S A. Then AS, FR, either meet the directrix in a common point R, or are parallel to the directrix. Also AP, FP either meet the directrix in a common point P, and consequently S P, A P, in a
12, 14. point P, or they are each parallel to the directrix. When AP, FP meet the directrix, because SR, FR, XX, have either one common concourse, or are mutually parallel to each other, and from two points P, P, in XX are drawn to SR the parallels P S, PA,

PA, and to FR the parallels pA, PF, therefore pS will be to pA as PA is to PF (LEM. 1.). But, because of the parallels, the angle SpA is equal to the angle APF; therefore the triangles ASp, FAP, are equiangular (6. e. 6.), and Sp being parallel to AP, SA is parallel to AF (29. e. 1.).—But if AP, FP, and their correspondents Sp, Ap, be parallel to the directrix, AS must in that case meet the directrix, because falling upon the parallels AP, Sp, it must meet XX which is parallel to them. AS, FR, therefore meeting XX in R, let them meet Fp, Ap, in G, G. Because Sp, Ap, are correspondent to AP, FP; SR is to AR (as RG is to RG (18. DEF.), viz. because of the parallels Ap, FP,) as AR is to FR. Wherefore in this case also, SA is parallel to AF, and consequently A is the point respondent to A, in which FR the focal respondent to AS meets the circle. But pA passes through this point A, therefore pA the focal correspondent does meet the circle in a point A respondent to A. FIG. 13.

Conversely. CASE 1. If FP the focal respondent to AP meet the circle in A, AP shall meet the section in a point respondent to A. 10, 11.

Every thing remaining as above, join SA, and draw FA parallel thereto, meeting AP in A. Because SA is the semidiameter of the generating circle, SA is to SE as Z is to FI (16. DEF.). But by the same reasoning as above, it is shewn that SA is to SE as AF is to AD; therefore ex æquo, AF is to AD as Z is to FI, and consequently the point A is in the section (1.). Wherefore AF being also parallel to SA, the points A, A, are respondent. (19. DEF.).

CASE 2. If the centre S be in the conic correspondent Sp, and Ap the focal correspondent meet the circle in a point A; I say that AP the conic respondent shall meet the section in a point respondent to A. 12, 13, 14.

 Join

FIG. Join SA, FA, and drawing SR the conic respondent to FA or FR, let it meet AP in A, and join AF. Then by the same reasoning, and in the same words as in CASE 2. above, it may be shewn that AF is parallel to SA. Therefore AR, FR, being respondent, and the centre S of the circle being in AR, the point A is that point in the section, which in CASE 1. of the Converse, is proved to be respondent to A, and therefore in this point AP does meet the section.

COR. 1. If AP meet the section again in another point B, then by the same reasoning it would be proved, that the respondent FP or focal correspondent AP meets the generating circle in another point B respondent to B.—And Conversely, if FP or AP meet the circle in another point B, the conic respondent AP will meet the section in another point B respondent to B.

COR. 2. If the right line joining any two points in a conic section be drawn, the right line joining the respondent points in a generating circle shall be the focal respondent or focal correspondent of the former, accordingly as the centre of the circle is in the right line joining the points in the section, or not. And Conversely.

COR. 3. If there be two respondents, and the conic correspondent in a generating circle, which passes through the centre, be drawn; then if the conic respondent meet the section, the right line drawn through the respondent point in the circle to meet the conic correspondent in the directrix, or parallel to the directrix, accordingly as the respondents meet the directrix, or are parallel to it, shall be the focal correspondent. And Conversely.

For the focal correspondent does pass through the respondent point in the circle, and it also passes through the point in which the conic correspondent meets the directrix, or it is parallel to the directrix. Both these attributes this right line possesses, and only one right line can possess them. Therefore the right line is the focal correspondent.

COR. 4. By the same reasoning, if the focal correspondent pertaining to a generating circle be given, the right line drawn from the

the centre of the circle to meet the focal correspondent in the directrix, or parallel to the directrix, accordingly as the focal correspondent meets or is parallel to the directrix, shall be the conic correspondent. FIG.

Cor. 5. A right line meets a conic section or the opposite hyperbolas in not more than two points; because if in more, the respondent focal line would meet a generating circle in more than two points, which is absurd.

Cor. 6. If round any point in a right line, which meeting a conic section, is parallel to the directrix, the generating circle be described; the segment of the right line intercepted between the centre of the circle and the section shall be equal to the segment of its focal respondent intercepted between the focus and the respondent point in the circle.

Let AB parallel to the directrix meet a conic section in A, and 11.
round any point S therein the generating circle be described. Draw Fa the focal respondent to AB, which will meet the circle in a point a respondent to A, and join FA, Sa. Because AB is parallel to the directrix, Fa will also be parallel thereto, and therefore parallel to AB, also FA, Sa, are parallel between themselves (19. Def.). Therefore AFaS is a parallelogram, and the opposite side SA is equal to Fa.

Cor. 7. If round any point in the diameter of a parabola the generating circle be described, the rectangle under the latus rectum of the axis, and the abscis of the diameter, viz. intercepted between the point and the vertex, is equal to the rectangle under the segments of any right line drawn through the focus to cut the circle.

Let AH be a diameter of the parabola, meeting the directrix 41.
in H, the section in A, and round any point S therein, viz. with the distance SH, describe the generating circle. Through the focus F draw HF meeting the circle again in a, and the axis FI, meeting the directrix in I, which produce to K, so that IK be equal to FI. Then FK is equal to the latus rectum of the axis

(2.

FIG. (2. DEF.); and I say, that the rectangle under FK, AS, is equal to the rectangle HFA.

Join HK, AF, and SA, and let SA meet the axis in L. The point A is respondent to A, by this PROP., and therefore AF is parallel to SA (19. DEF.), and AFLS being a parallelogram, the opposite sides AS, FL, are equal between themselves. Because FI is equal to IK, HI common, and the angles at I are right, the angle HKI or HKL is equal (to the angle HFI (4. e. 1.), viz. to the alternate angle AHS, viz.) to the angle HAS or HAL. Wherefore the four points H, K, A, L, are in a circle, and the rectangle HFA is equal (to the rectangle KFL, viz.) to the rectangle under FK, AS.

SCHOLIUM. LOCUS.

21, 22. If there be a circle, whose centre is A, XX a right line in the same plane, and F be any point, other than A, without the right line XX. Then if two rulers AP, FP, revolve about A, F, carrying their intersection P constantly in the right line XX, while two other rulers AB, FB, revolve also about A, F, and the one AB following the intersection B of the ruler FP with the circumference of the circle, the other FB preserves a constant parallelism to AB; the intersection B of this last ruler FB with the ruler AP, will by its motion describe a conic section, whose focus is F, and directrix XX.

22. The point P passing to an infinite distance, the rulers AP, FP, at the same time become parallel to XX.

This Locus is demonstrated in the preceding PROP., and for this reason, every circle thus related to a conic section, is in the 16. DEF. called a GENERATING CIRCLE.

As the right line XX touches, falls without, or within the circle; or as the perpendicular distance of the centre of the circle from the directrix is equal to, greater, or less than, the semidiameter

meter of the circle, the section is accordingly a parabola, an ellipse, or hyperbola. FIG.

As the circle is divided into two segments by the directrix, in the case of the hyperbola, if the centre of the circle be towards the parts of the directrix opposite to the focus, every intersection b in that segment, which is on the parts of the directrix opposite to the focus, contributes to the description of the opposite hyperbola, while during the passage of the intersection a through that segment, which is on the same parts of the directrix with the focus, the hyperbola on the same side with the focus is described. And E contra, if the centre of the circle be towards the same parts of the directrix with the focus.

PROP. 4.

If two right lines, pertaining to a conic section, meet each other in a point not in the section, their focal respondents or focal correspondents, pertaining to a generating circle, shall meet each other in a correspondent point. And Conversely.

Let AP, BR, two right lines pertaining to a conic section, meet each other in a point Q, which is not in the section; and ap, br, be their focal respondents or focal correspondents, pertaining to a generating circle, whose centre is S; I say, that ap, br, shall meet each other in a point correspondent to Q. 15, 16.

1. When AP, BR, both meet the directrix.

Let them meet it in P, R, and therefore ap, br, meet it in p, r. Join FP, FR, Sp, Sr. Then Sp, Sr, are the conic correspondents to AP, BR (4. Cor. 3.). Because PF, RF, meet in F, ap, br, which are parallel to them, will meet. Let them meet in q, and join Sq, FQ. On account of the parallels, the triangles FPR, qpr, and QPR, Spr, are equiangular. There- 15.

FIG. fore FR is to qr (as PR is to pr, viz.) as QR is to Sr. But FR, QR, are alſo parallel to qr, Sr; therefore the angle FRQ is equal to the angle qrS, and the triangles QFR, Sqr, are equiangular (6. e. 6.); and the homologous ſides FR, qr, being parallel, the homologous ſides FQ, Sq, will alſo be parallel. Wherefore the point q, the concourſe of ap, br, is correſpondent to Q, the concourſe of AP, BR (20 Def.).

16. 2. When one of the conic lines, as AP, is parallel to the directrix.

Becauſe AP is parallel to the directrix, FP, Sp, ap, are alſo parallel thereto. Other things therefore remaining the ſame, let BR meet FP in G, and Sr meet ap in g. Becauſe Sp, ap, are correſpondent to AP, FP, and each parallel to the directrix, QR is to Sr as GR is to gr (18. Def.). But, becauſe of the parallels, the triangles FRG, qrg, are equiangular; therefore FR is to qr, as GR is to gr; and ex æquo, QR is to Sr as FR is to qr. Wherefore, as in Case 1., FQ will be parallel to Sq, and the point q, the concourſe of ap, br, will be correſpondent to the point Q, the concourſe of AP, BR (20. Def.).

Converſely. If two focal correſpondents ap, br, meeting a point q, their reſpondents AP, BR, ſhall meet in a correſpondent point.

The ſame things remaining, the demonſtration will be the ſame, and in the ſame terms.

Cor. 1. If two right lines, pertaining to a conic ſection, meet each other, and in the focal correſpondent to one of them, pertaining to a generating circle, the point be aſſumed, which is correſpondent to the point of concourſe of the conic lines; the focal correſpondent to the other ſhall paſs through this correſpondent point.

17. Let AB, AP, pertaining to a conic ſection, meet in A, and in ap the focal correſpondent to AP, and pertaining to a generating circle, whoſe centre is S, let the point a be aſſumed correſpondent to A; I ſay, that the focal correſpondent to AB ſhall paſs through a.

If

If not, draw ab the focal correſpondent to AB, meeting AP in the point a, other than A, and join Sa, SA, AF. Becauſe A, A, are correſpondent points, SA is parallel to AF (20. DEF.); and becauſe AP, AB, meet in A, and AP, ab, their focal correſpondents meet in a, therefore by this PROP. Sa ſhall be parallel alſo to AF. Wherefore SA is parallel to Sa; viz. two right lines, parallel to each other, meet in a point S, which is abſurd. Wherefore no right line but that which paſſes through A can be the focal correſpondent to AB. FIG.

COR. 2. Right lines, joining correſpondent points, are themſelves correſpondent.

If A, B, be two points in the right lines AP, BR, pertaining to a conic ſection, and A, B, two correſpondent points in AP, BR, their focal correſpondents, pertaining to a generating circle. I ſay, that AB ſhall be the focal correſpondent to AB. 17, 18.

For the focal correſpondent to AB paſſes through each of the points A, B, by the preceding corollary, and conſequently is the right line AB itſelf.

PROP. 5.

If from a point in the plane of a conic ſection and from the focus two right lines parallel to each other be drawn to meet the directrix; then as the parallel drawn from the ſaid point has to the focal diſtance of the point, the ſame ratio, a greater, or a leſs than that, which the parallel drawn from the focus has to the ſemi-latus rectum of the focal axis, the point ſhall be IN, WITHIN, or WITHOUT the ſection.

FIG. And converſely, as the point is in, within, or without the ſection, the ratio ſhall be the ſame, greater, or leſs.

24. The firſt part of the PROP. which declares the point to be in the ſection, if the ratio be the ſame, together with its converſe, is already demonſtrated in PROP. 1.

21, 22, 23. Let A then be a point in the plane of a conic ſection, whoſe focus is F, directrix XX, and ſemi-parameter of the focal axis Z. If from A, F, be drawn to the directrix any two parallels AH, FG, and AF be joined; I ſay, that as the ratio of AH to AF is is greater or leſs than that of FG to Z, the point A ſhall be within or without the ſection.

If the ratio be greater, the point ſhall be within.

21, 22. Round A deſcribe the generating circle (15. COR. DEF.), draw any right line AP through A, and AO a ſemidiameter of the circle. AH is to AO as FG is to Z (16. DEF.), and AH is to AF in a greater ratio than FG is to Z; therefore ex æquo, AH has to AF a greater ratio than AH has to AO. Wherefore AF is leſs than AO (10. e. 5.), viz. the focus F is within the circle, and conſequently every right line drawn through F will meet the circle. Draw FP reſpondent to AP, meeting the circle in B, the conic correſpondent AP will meet the ſection in a point B reſpondent to B (3.), as for the ſame reaſon, will every right line drawn through A meet the ſection. The point A is therefore within the ſection (7. DEF.).

If the ratio be leſs, the point ſhall be without the ſection.

23. Other things remaining the ſame, let the ratio of AH to AF be leſs than that of FG to Z; I ſay the point A ſhall be without the ſection. For, as before, AH is to AO as FG is to Z; therefore ex æquo, AH has to AF a leſs ratio than AH has to AO, and conſequently AF is greater than AO (10. e. 5.), and the point F is without the circle. Wherefore a right line may be

drawn

drawn through F, which shall fall entirely without the circle. Let FP be so drawn, and AP respondent to it. Because AH has to AF a less ratio than FG has to Z, the point A is not in the section (1.). From any other point B in AP draw BF, and Ab parallel thereto, meeting FP in b, also draw BK parallel to AH or FG, meeting the directrix in K. Because AP, FP, and the directrix either have one common concourse in P, or are parallel each to each other (17. DEF.), and from two points A, B in AP are drawn to the directrix the parallels AH, BK, and to FP the parallels Ab, BF, therefore BK will be to BF as AH is to Ab (LEM. 1.). But as FP falls entirely without the circle, Ab will be greater than AO, and consequently AH has to Ab a less ratio (than AH has to AO, viz.) than FG has to Z (16. DEF.). Wherefore ex æquo, BK has to BF a less ratio than FG has to Z, and consequently the point B is not in the section (1.). Nor for the same reason is any point whatever in AP in the section. Through the point A therefore is drawn a right line AP, which, however extended, no where meets the section, and consequently the point A is without the section (7. DEF.).

FIG.

Conversely. If the point A be within the section, the ratio of AH to AF shall be greater than that of FG to Z. For it is absurd to say, that it is either the same or less, as the point A would then be either in or without the section.—And if the point A be without the section, the ratio must be less, because if it were either the same or greater, the point would be either in or within the section.

COR. 1. A point in the plane of a conic section is in, within, or without the section, accordingly as its distance from the directrix has to its distance from the focus the same ratio, a greater, or a less than that, which the distance of any point in the section from the directrix has to its focal distance, or which the distance of the centre of any generating circle from the directrix has to the semidiameter of the circle.

FIG. For these ratios are the same with that in the Prop., whose equality, excess or defect determines a point to be in, within, or without the section.

Note. The distances from the directrix, in the same manner as in the 16. Def., are measured either perpendicularly, or by any parallels drawn from the points to the directrix.

Cor. 2. As the centre of a generating circle is in, within or without the section, the focus is in, within or without the circle. And conversely, as the focus is in, within or without the circle, the centre of the circle is in, within or without the section.

This appeared in the demonstration.

Cor. 3. As the semidiameter of a generating circle is equal to, greater or less than, the focal distance of its centre, the centre is in, within, or without the section. And conversely.

The same things remaining as in the Prop., as AO is equal to, greater or less than AF, the ratio of AH to AF is equal to, greater or less than, that of AH to AO; and therefore accordingly the point A is in, within, or without the section (1. Cor.). And in like manner conversely.

PROP. 6.

A right line parallel to the diameters of a parabola, or to an assymptote of an hyperbola, but towards the same parts of the assymptote with the hyperbola, meets either section in one point only, and towards the parts below the concourse falls within the section.

24, 25, 26. Let GE be a parabola or hyperbola, F the focus, XX the directrix, CM, Cm, the assymptotes of the hyperbola, meeting the directrix in M, m, and RP a parallel to the axis of the parabola, or to either assymptote CM of the hyperbola, but towards the same

parts of the aſſymptote with the hyperbola GE. I ſay, that RP meets the ſection GE. FIG.

Let RP meet the directrix in P, and join PF. Draw FD making with FP the angle PFD equal to the angle FPR. In the parabola, becauſe PR is perpendicular to XX, the angle FPR is but a part of a right angle, and conſequently FPR, PFD, are together leſs than two right angles. In the hyperbola, join FM meeting PR in H. Becauſe FM is perpendicular to CM (19. COR. DEF.), viz. to PR, the angle FHP is a right angle. Wherefore the angle FPR is leſs than a right angle, and therefore in the hyperbola alſo, the equal angles FPR, PFD, are together leſs than two right angles. In both ſections therefore PR, FD, will meet (12. DEF. e. 1.), and in the caſe of the hyperbola, towards the parts of the directrix with the ſection, towards which parts the angles are leſs. Let them meet in A. Becauſe AP is equal to the ſemidiameter of the generating circle deſcribed round A (6. & 8. COR. DEF.), and on account of the equal angles PFA, APF, AP is equal to AF, the point A will be in the ſection (3. COR. 5.). Wherefore RP does meet the ſection.

I ſay farther, that below the concourſe A, viz. towards the parts of A remote from the directrix, PR falls within the ſection.

Let S be any point in PR below the concourſe A, viz. on the parts of A oppoſite to P, round which point S deſcribe the generating circle, and draw SB parallel to AF, meeting PF in B. Then AP being equal to AF, SP will be equal to SB, and becauſe SP is equal to the ſemidiameter of the generating circle deſcribed round S (6. & 8. COR. DEF.), the circumference will paſs through P, B. Wherefore the point A being between the terms P, S, the point F will be between the terms P, B, viz. the focus F is within the circle, and conſequently the centre S is within the ſection (2. COR. 5.). For the ſame reaſon is every point in PR, below the concourſe A, within the ſection.

In the ſame manner if a point s were aſſumed in PR above the concourſe A, viz. on the parts of A towards the directrix, and a generating

FIG. generating circle were described round s, it would be proved that the focus F is without the circle, and consequently that the point s is without the section.

The right line RP does therefore meet each section in one point only, and below the concourse fall within the section.

COR. 1. If a right line touch the parabola or hyperbola, or meet either section in two points, or meet the opposite hyperbolas, the right line will meet the diameter of the parabola, or the assymptotes of the hyperbola.

For if not, the right line must be parallel to the diameters of the parabola, or to an assymptote of the hyperbola, and consequently would meet either section in one point only, and fall within it, which is contrary to the position.

COR. 2. Hence it appears, that the parabola and hyperbola continually recede from the axis, and that every successive point in each section is at a greater distance from the axis, or that a point remote from the vertex is at a greater distance from the axis, than a point which is nearer to the vertex.

COR. 3. From the demonstration it appears, that if from any point in either of the opposite hyperbolas two right lines be drawn, one to the focus, the other parallel to an assymptote and meeting the directrix, these two right lines shall be equal between themselves; and therefore, that the parabola and hyperbola agree in this, that a point in either section is equidistant from the focus and from the directrix; save that in the parabola the distance from the directrix is measured by a perpendicular, in the hyperbola by a parallel to an assymptote.

SCHOLIUM LOCUS.

If a ruler SP move with its term P always in the fixed right line XX, preserving a constant parallelism to itself in its first situation, and from a fixed point F, without XX be constantly drawn

to SP the right line FA, so that FA be equal to AP, the line described by the motion of the point A shall be a parabola or hyperbola, accordingly as the angle SPX is right or oblique, of which section XX shall be the directrix, and the point F the focus. FIG.

PROP. 7.

A transverse diameter, or a right line parallel to a transverse diameter, meets each of the opposite hyperbolas in one point only, and passes within each section.

Let DH be either a transverse diameter, or parallel to a transverse diameter CO; I say, that it meets each of the opposite hyperbolas in one point only. 27, 28.

Let CM, Cm, be the assymptotes meeting the directrix in M, m, and round any point S in DH describe the generating circle, meeting the directrix in K, k. Join SK, Sk, which are parallel to CM, Cm (8. Cor. Def.). Because DH is either a transverse diameter or parallel to one, it falls within the angle KSk (15. Def.), viz. it meets the directrix in a point P between the terms K, k. Wherefore the focal respondent FP will meet each segment into which the circumference of the circle is divided by the directrix, as in the points a, b; and the conic respondent DH will meet each hyperbola, the one in a point A respondent to a, the other in a point respondent to the concourse of FP with the alternate segment of the circle (1. Cor. 3.), and it can meet each in that one point only (5. Cor. 3.).

It is farther asserted, that after each concourse the right line DH falls within each section. Around any point S therein, below either concourse, as A, describe the generating circle. The focal respondent FP will meet this circle in a point a respondent

to

FIG. to A (3.), and AF, AS, being joined will be parallel to each other (19. Def.). But the point A being between the terms S, P, the focus F will be between the terms A, P, viz. within the circle, because the point P is itself within the circle. Wherefore the point S is within the section (2. Cor. 5.), as for the same reason is every point in DH, below the concourse A.

PROP. 8.

Every right line drawn through a point within the ellipse; every right line drawn through a point within the parabola, which is not parallel to the diameters; every right line drawn through a point within the hyperbola, which is neither parallel to an assymptote, nor to a transverse diameter, meets the section in two points.

10, 11. Let SP be drawn through a point S within a conic section, but not parallel to the diameters of a parabola, nor to an assymptote nor to a transverse diameter of an hyperbola; I say that SP will meet the section in two points.

Round S describe a generating circle, and draw FP respondent to SP. Because S is within the section, the focus F will be within the circle (2. Cor. 5.), and therefore FP will meet the circle in two points A, B. But the points A, B, will be both on the same side of the directrix with S; in the hyperbola, because SP is parallel to a diameter secunda; in the ellipse, because the generating circle falls wholly on the same side of the directrix with the section itself; and in the parabola, because the circle meets the directrix in one point only, through which point FP by supposition, does not pass. Wherefore, FP meeting the circle in two points, both

on

on the same side of the directrix with S, viz. with the section, SP will meet the section in two respondent points A, B (1. COR. 3.). FIG.

PROP. 9.

If a right line touch a conic section, and any generating circle be described, then accordingly as the centre of the circle is in the touching line or not, the focal respondent or focal correspondent shall touch the circle in a respondent point.

And conversely, if a right line touch a generating circle, the conic respondent shall touch the section in the respondent point.

Let AP touch a conic section in A, F being the focus, and XX the directrix of the section. And

1. Let the generating circle be described about the point A itself. I say, that FP, respondent to AP, shall touch the circle.— For FP does meet the circle in the point F, respondent to A (3. or 2. COR. 5.), but I say farther, that it touches the circle in F. Let G be any other point in FP, join AG, AF, draw FS parallel to AG meeting AP in S, and AD, SE parallel to each other, meeting the directrix in D, E. Because AP touches the section in A, the point S is without the section (8. DEF.), and therefore SE is to SF in a less ratio than AD is to AF (1. COR. 5.). But because AP, FP, and the directrix have either one common concourse, or are parallel between themselves, and from two points A, S, in AP are drawn to the directrix the parallels AD, SE, and to FP the parallels AG, SF, therefore SE is to SF as AD is to AG (LEM. 1.). Wherefore ex æquo, AD has to AG a less ratio than AD has to AF, and consequently AG is greater 29, 30. 31.

FIG. than AF (10. e. 5), viz. the point G is without the circle. For the ſame reaſon is every point in FP, unleſs the point F alone, without the circle; that is, FP touches the circle in F.

Converſely, if FP touch the circle in F, AP the reſpondent to FP ſhall touch the ſection in a point reſpondent to F. For it does meet the ſection in a point reſpondent to F (3.), and A being in the ſection, and the centre of the circle which paſſes through F, it is the point reſpondent to F. Wherefore let S be any other point in AP, and every thing be drawn as in the preceding. Becauſe FP touches the circle in F, the point G is without the circle, viz. AG is greater than AF, and therefore AD has to AG a leſs ratio than AD has to AF (8. e. 5.). But for the ſame reaſon as above, SE is to SF as AD is to AG, and therefore, ex æquo, SE has to SF a leſs ratio than AD has to AF. Wherefore the point S is without the ſection (1. COR. 5.), as for the ſame reaſon is every point in AP, unleſs the point A alone; viz. AP touches the ſection in A.

Let the generating circle be deſcribed about any other point in AP, viz. S; I ſay, that if AP touch the ſection in A, FP reſpondent to AP ſhall touch the circle in a point reſpondent to A.

For it does meet the circle in a reſpondent point (3.), which point let be A, and join SA, FA. Then, becauſe A, A, are reſpondent points, SA is parallel to AF (19. DEF.); but by the preceding, FP touches in F the generating circle deſcribed round A, and therefore is perpendicular to AF. Wherefore PF is alſo perpendicular to SA, and conſequently touches the circle deſcribed round S in the point A.

Converſely. If FP touch the circle in A, its reſpondent AP will touch the ſection in a point reſpondent to A.—For it does meet the ſection in a reſpondent point, as A, and SA, FA, being joined, will be parallel (19. DEF.). But, becauſe FP, touching the circle in A, is perpendicular to SA, therefore FP will alſo be perpendicular to AF, and conſequently AP will touch rhe ſection in A, by the 1. CASE of this.

3. Let

3. Let the generating circle be described about any point S, which is not in AP. If AP touch the section in A, the focal correspondent shall touch the circle in a respondent point. FIG. 32, 33, 34, 35.

Draw FP respondent to AP, and Sp, ap, correspondents to AP, FP. Then pa meets the circle in a point respondent to A (3.), as a, and Sa, FA, being joined, will be parallel (19. Def.). But, because AP touches the section in A, FP will be perpendicular to AF, by Case 1., therefore, on account of the parallels, pa will be perpendicular to Sa, and consequently will touch the circle in a. 36, 37.

Conversely. If the focal correspondent pa touch the circle in a, the conic respondent AP shall touch the section in a point respondent to a.

For AP does meet the section in a respondent point, as A (3.), and every thing being drawn as before, because pa touches the circle in a, it is perpendicular to Sa. Therefore on account of the parallels, FP is also perpendicular to AF, and consequently, by Case 1., AP touches the section in A.

Cor. 1. If a right line touch a conic section, and meet the directrix, the right lines drawn to the focus from the point of contact and from the point of concourse with the directrix, shall be at right angles to each other.

And conversely, if two right lines, meeting in the focus, be at right angles to each other, and one of them meet the section, the other the directrix, the right line joining the two points of concourse with the section and directrix, shall touch the section.

Cor. 2. Only one right line can touch a conic section in the same point. For, if more, then more than one respondent line would touch a circle in the same respondent point, which is absurd.

Cor. 3. A right line touching a conic section in the vertex of the focal axis itself, is parallel to the directrix. And if a right line, touching a conic section, be parallel to the directrix, the point of contact shall be a vertex of the focal axis.

FIG. Let AP touch a conic section in a vertex A of the focal axis FA;
31. I say, that AP is parallel to the directrix.—Draw FP the focal respondent to AP; then, by the PROP., FP is perpendicular to FA, viz. is parallel to the directrix, and consequently its respondent AP will also be parallel to the directrix.

And conversely, if AP parallel to the directrix touch the section in A, the point A shall be a vertex of the focal axis.—For AP, being parallel to the directrix, must meet the axis, and meeting the section also, if it meet the axis in any other point than a vertex, it must meet it below the vertex, that is, it will fall within the section, (by the same reasoning as in PROP. 6.), which is absurd. The point A is therefore the vertex itself.

COR. 4. Hence therefore a right line touching a conic section in any other point than a vertex of the focal axis, meets the directrix.

COR. 5. If a right line passing through the focus meet the same section or opposite sections in two points, the right lines touching the section in these points shall have their concourse in the directrix, and the right line joining their concourse, and the focus shall be perpendicular to the right line cutting the section or sections.

29, 30. Let AH passing through F the focus, meet the same section or the opposite sections in A, H; I say, that the tangents at A, H, shall have their concourse in the directrix.

Draw FP perpendicular to AH, meeting the directrix in P, and join AP, HP. Then AP, HP, each touch the section (1. COR.), and by construction, their concourse P is in the directrix, and PF is perpendicular to AH.

COR. 6. If a right line passing through the focus, and parallel to the directrix, meet a conic section in two points, the right lines touching the section in these points shall meet in the concourse of the focal axis with the directrix.

This is only a case of the preceding corollary.

COR. 7. A right line touching a parabola in the vertex of a diameter, bisects the angle comprehended by the diameter and the right

right line drawn from the vertex to the focus, and is perpendicular to the right line which joins the focus and concourse of the diameter with the directrix. FIG.

The same things remaining, let a diameter of the parabola pass through the point of contact A, and join Fb. Because AFP, 32.
AbP, are each right angles, and AF is equal to Ab, while AP is common, the side FP will be equal to the side bP (47. e. 1.), and consequently the angle FAP be equal to the angle bAP (8. e. 1.). The right line AP therefore, which bisects the angle bAF of the isosceles triangle bAF, will also bisect the base Fb, and be perpendicular to it (4. e. 1.).

COR. 8. Hence therefore, the segment of a diameter of the parabola, intercepted between the directrix and a right line drawn through the focus parallel to the tangent at the vertex, is bisected in the vertex.

Draw FD parallel to AP, meeting Ab in D. Because bF is 32.
perpendicular to AP, it will also be perpendicular to FD. Therefore the angle bFD is in a semicircle, whose diameter is bD (31. e. 3.), and consequently Ab being equal to AF, the centre of this semicircle will be in the point A. Wherefore the diameter bD is bisected in A.

PROP. 10. PROBLEM.

The focus, the directrix, and a point not within the section being given, it is required to draw a right line through the said point, which shall touch the section.

1. When the point given is in the section.

Let the point given in the section be A, F be the focus, and 29, 30.
XX be the directrix. Join AF, and draw FP perpendicular thereto. If A be the vertex of the focal axis, FP will be parallel 31.
to

FIG. to the directrix, and AP parallel to FP will touch the section in A (3. Cor. 9.). But if A be not a vertex of the axis, AF will not be the axis, and therefore FP perpendicular thereto will not be parallel to the directrix, but will meet it, as in P. Join AP, which will touch the section in A (1. Cor. 9.).

2. When the point given is without the section, but is a point in the directrix.

29, 30. Let P be the point given in the directrix; join PF, and draw FA perpendicular thereto, meeting the section in A (1. Def.) Join AP, which will touch the section (1. Cor. 9.).

3. When the point given is without the section, but not in the directrix.

29, 30. 31. Let S be the given point, round which describe the generating circle. Because the point S is without the section, the focus F is without the circle (2. Cor. 5.). A right line therefore may be drawn from F to touch the circle (17. e. 3.); let FP be so drawn, touching the circle in a, draw SP respondent to FP, and joining Sa, draw FA, parallel to Sa, meeting SP in A. The point A is respondent to a, and therefore in the section (3.); and because FP touches the circle in a, SP will touch the section in A (9.).

Cor. 1. In the parabola and ellipse two right lines may always be drawn from any point without either section to touch it.

For if, as in Case 2., the given point be in the directrix, because FP meeting the directrix, is not parallel to the directrix, therefore AF perpendicular to FP is not perpendicular to the directrix, and therefore must meet the section in two points (8.). Wherefore the right lines drawn to either of them from the given point in the directrix will touch the section. But if the given point be not in the directrix, from the construction of Case 3., because from F may always be drawn two right lines to touch the generating circle described round S, neither of which points can be in the directrix, there will always be two correspondent points in either section, and therefore the right lines drawn thereto from the given point will touch the section.

Cor.

COR. 2. In the hyperbola, two right lines may be drawn from a point without each of the hyperbolas to touch either the ſame, or to touch the oppoſite ſections, unleſs the point given be in an aſſymptote, when only one right line can be drawn to touch one ſection. FIG.

If the point given be M, the concourſe of an aſſymptote with the directrix, then becauſe FM is perpendicular to the aſſymptote CM (18. COR. DEF.), a perpendicular to FM at the point F, which CASE 2. requires, would be parallel to CM, and therefore meeting only one of the ſections in one point only (6.), one of the points to which the touching lines would be drawn vaniſhes, and to that point alone in one of the ſections, which the perpendicular to FM meets, one line touching that ſection can be drawn.—If the point given in the aſſymptote be any other point S, becauſe the generating circle deſcribed round S paſſes through M (8. COR. DEF.), FM would be one of the tangents to this circle, which CASE 3. requires to be drawn, and the point M would become the ſame with both A and P, and SP would be the aſſymptote itſelf. Wherefore the parallel to SA drawn through F would be parallel to SM, and not meeting SM or SP, the point A would vaniſh. The other tangent drawn from F to touch the circle would then alone determine a point in the ſection, to which a right line can be drawn from S to touch the ſection. 5.

But when the point given is in the directrix, and is any other than M or M, the right line drawn thereto from F as FP will not be perpendicular to an aſſymptote, and therefore FA perpendicular to FP will not be parallel to an aſſymptote, and conſequently will meet the ſame or the oppoſite ſections in two points (8. & 7.), to which if right lines be drawn from P, each of them will touch the ſection.—While if the given point be neither in the directrix nor in an aſſymptote, it is comprehended under CASE 3., and univerſally admits of two right lines being drawn to touch the hyperbola or hyperbolas, as well as in the ellipſe and parabola, and for the ſame reaſons. 29.

NOTE.

FIG. NOTE. If the given point be in an aſſymptote of the hyperbola, the ſingle touching line drawn therefrom will touch that hyperbola which is towards the ſame parts of the centre with the point given. If the given point be within either angle of the aſſymptotes which comprehend the hyperbolas, the two right lines drawn from the point will touch that ſection which is comprehended in the ſame angle as the point, but if the point given be within either of the adjacent angles, the two right lines drawn from the given point will touch the oppoſite hyperbolas.

This would be eaſily illuſtrated, if it were neceſſary, from the 7. & 8. PROP., and in CASE 3. from the focal tangents to the circle touching the ſame ſegment, or the oppoſite ſegments into which the circle is divided by the directrix.

PROP. 11. PROBLEM.

The focus, directrix, and principal ſemi-parameter of a conic ſection being given, it is required to draw a right line touching the ſection, which ſhall be parallel to a right line given in poſition.

32, 33. ANALYSIS. Suppoſe it to be done; viz. that F being the
34, 35. focus, XX the directrix of a conic ſection, Z the principal ſemi-
36, 37. parameter, and Sp a right line given in poſition, there is drawn a right line AP parallel to Sp, which touches the ſection in A.

Round any point S in Sp deſcribe the generating circle. This circle is given (15. COR. DEF.). Join AF, and draw Sa parallel thereto, meeting the circle in a, reſpondent to A in the ſection (3.). Draw FP the focal reſpondent to AP, and ap parallel to FP. Then Sp, ap are correſpondent to AP, FP; and if Sp be parallel to the directrix, the reſpondents and correſpondents will all be parallel to the directrix; but if Sp meet the directrix, AP, FP,

FP, will meet the directrix in one and the ſame point, as P, and FIG.
SP, AP, in one and the ſame point, as P (17. & 18. DEF.). Becauſe AP touches the ſection in A, AP touches the circle in A (9.), therefore SA is perpendicular to AP (18. e. 3.), and conſequently when AP is parallel to the directrix, SA is perpendicular to the directrix, and will therefore be given, and each of the points A, B, in which it meets the circle, will be given. But if SP meet the directrix, the point P is given, and conſequently the point A, in which AP touches the circle (94. DATA.). Therefore whether SP be parallel to the directrix or meet it, the point A is given.— 35, 36.
If SP be parallel to the directrix, join AF, SA, which will meet 37.
the directrix in a common point L, becauſe A, A, are reſpondent points (3.). But the points A, S, F, being given, as alſo the directrix in poſition, AF is given in poſition, the point L is given, and the poſition of SL. Alſo, SA being given in poſition, FA, which is parallel thereto, is given in poſition, and conſequently the point A is given.—If SP meet the directrix in the given point P, 32, 33.
then SP, AP, SA, are each given in poſition and magnitude, and 34.
therefore the point F being given, FP, FA, parallel to AP, SA, will be given in poſition, the point P will be given, PA will be given in poſition, and the point A the concourſe of PA with FA. Wherefore the poſition of the right line which being parallel to SP, is required to touch the ſection, and the point in which it touches it, are given.

LIMITS. If SP, and conſequently AP, be parallel to the directrix, then univerſally two right lines AP, BP, each parallel to the 35, 36. 37.
directrix, may be drawn to touch the circle, and they will be at the extremities of a diameter AB perpendicular to each, and therefore perpendicular to the directrix. In the ellipſe and hyperbola, 35, 36.
theſe two points A, B, will always fall without the directrix, and in the hyperbola one of theſe points will be in one, the other in the other, of thoſe ſegments into which every generating circle of the hyperbola is divided. Wherefore in the ellipſe and hyperbola, when

FIG. the right line given in position is parallel to the directrix, the problem always admits of two solutions, and there will be two points A, B, in each section, respondent to A, B, in the circle, through which if AP, BR, be drawn, parallel to SP or to the directrix, they will each touch the section. But, because AF, BF, are respectively parallel to SA, SB, viz. perpendicular to the directrix,
37. the points A, B, will be the vertices of the focal axis.—Also, in the parabola, AB being for the same reason perpendicular to the directrix, one of the points, as B, will be that in which the directrix touches the circle, which point therefore can have no respondent point in the section. Wherefore in the parabola, one and only one right line may be drawn parallel to the directrix to touch the section, and for the same reason, as in the ellipse and hyperbola, the point in which it touches the parabola will be the vertex of the axis.

32, 33. But if SP meet the directrix, it must be in some point without
34. the circle. For from a point within the circle is impossible, and from a point in the circle only the single one touching it in the point itself can be drawn. But as this point must be in the directrix, in the case of the ellipse or hyperbola, and therefore can have no point respondent to it in the section, the problem would in this instance be impossible. Wherefore the right line SP must not meet the directrix within the circle, which can only happen in the hyperbola, when it is parallel to a transverse diameter; nor must it meet the directrix in the point in which the generating circle meets the directrix, which can only be in the parabola, when SP is perpendicular to the directrix, and therefore meets the directrix and circle in their point of contact, or in the hyperbola, when being parallel to an assymptote it meets the directrix in either of the points in which the circle cuts the directrix. The right line given in position must not therefore be parallel to the diameters of a parabola, nor to a transverse diameter, nor to an assymptote of the hyperbola, as in each of these cases the problem would be impossible.—In every other, it will meet the directrix of the

the parabola and hyperbola without the circle. While in the ellipse, it meets the directrix always without the circle, whatever be the position of the right line given, because every generating circle of the ellipse falls entirely without the directrix. With these limits therefore in the parabola and hyperbola, and universally in the ellipse, two right lines may be drawn from the point P to touch the circle, and the points of contact A, B, being each without the directrix, two points A, B, in each section, will be respondent to them, and therefore the problem will admit of two solutions. But in the hyperbola, because the points of contact A, B, in the circle will be in different segments, the respondent points of contact A, B, will be, one in one hyperbola, the other in the opposite. In the parabola, because every generating circle touches the directrix, therefore one of the right lines drawn from a point in the directrix to touch the circle will be the directrix itself. But as this point of contact B can have no respondent point in the section, the problem, in the case of the parabola, never admits of more than one solution. FIG.

The Limits therefore are these,

1. In the case of the PARABOLA, the right line given in position must not be parallel to the diameters. In every other position, the problem admits of one and of one solution only.

2. In the case of the HYPERBOLA, the right line must not be parallel to a transverse diameter, nor to an assymptote; in every other position, the problem admits of two solutions.

3. In the case of the ELLIPSE, whatever be the position of the right line given, the problem admits of two solutions.

COMPOSITION. Let F be the focus, XX the directrix, Z the principal semiparameter, and SP the right line given in position, round any point S in which describe the generating circle. If SP be parallel to the directrix, draw ASB the diameter of the circle perpendicular to the directrix. This diameter will meet the directrix of the parabola in one of its extremities B, while in the ellipse 35, 36. 37.

FIG. and hyperbola both its extremities will be without the directrix. From the other extremity A in the parabola, but in the other two sections from either extremity, as A, of the diameter AB, draw AF, meeting the directrix in L, and joining SL, draw AF parallel to AB, meeting SL in A. This point A will be a point in the section, and respondent to A in the circle (3.). Draw AP, AP, parallel to SP, viz. to the directrix. Because AP touches the circle in A (COR. 16. e. 3.), AP will touch the section in A (9.).

32, 33. But if SP meet the directrix, as in P; the point P will be with-
34. out the circle, because it is required from the limits, that SP be not parallel to the diameters of a parabola, nor to a transverse diameter, nor to an assymptote of an hyperbola; while without any limitation in the ellipse, every right line drawn through a point within a generating circle must, if it meet the directrix at all, meet it without the circle. From the point P, draw PA to touch the circle (17. e. 3.), but this must not, in the case of the parabola, be the directrix itself, while in the ellipse and hyperbola either of the right lines which are drawn from P to touch the circle, equally serve the purpose of the problem. Join SA, draw FP parallel to AP, meeting the directrix in P, and PA, FA, parallel to SP, SA, meeting each other in A. The point A is respondent to A, and is in the section (19. DEF. & 3.). Wherefore because AP touches the circle in A, AP will touch the section in A (9.).

COR. 1. If two right lines parallel to the directrix touch an ellipse or hyperbola, they touch each section in the vertices of the focal axis.

COR. 2. Only one right line parallel to a right line given in position can touch a parabola.

COR. 3. In the hyperbola, two right lines parallel to a right line given in position may be drawn to touch the opposite sections, provided the right line given in position be neither parallel to a transverse diameter nor to an assymptote.

COR.

FIG.

COR. 4. If two right lines, parallel to each other, but not parallel to the directrix, touch an ellipse or hyperbola, the right lines touching any generating circle in the respondent points will have their concourse in the directrix. And Conversely, if two right lines touching a generating circle of the ellipse or hyperbola, meet each other in the directrix, the right lines touching the section in the respondent points shall be parallel to each other.

This is no more than stating what appears in the problem, and enters into its demonstration. For A, B, are the points in the circle 32, 33. respondent to A, B, in the section; and AP, BP, touching the 34. circle in A, B, have their concourse in the directrix (12. COR. DEF.). Also, from a point P in the directrix, without the circle, PA, PB, being drawn to touch the circle in A, B, the right lines AP, BR, touching the section or sections in the respondent points A, B, are the conic correspondents to AP, BP, and therefore are parallel between themselves (12. COR. DEF.).

PROP. 12.

The right lines touching an ellipse or the opposite hyperbolas in the vertices of a diameter are parallel to each other; and if two right lines parallel to each other do touch an ellipse or the opposite hyperbolas, the two points of contact shall be diametrically opposite, viz. the right line joining them shall pass through the centre of the section.

CASE I. If the diameter be the focal axis, then the right lines touching the sections in the vertices are parallel to the directrix (3. COR. 9.), and therefore parallel to each other.

And Conversely, if the touching right lines, being parallel to each other, are parallel also to the directrix, they do touch each section.

FIG. ſection in the vertices of the focal axis (3. COR. 9.), and therefore the points of contact are diametrically oppoſite to each other.

38, 39. CASE 2. If AP, BR, touch an ellipſe or hyperbola in the
40. vertices A, B, of any other diameter ACB, which is not a diameter ſecunda of the hyperbola; I ſay, that AP ſhall be parallel to BR.

Becauſe A, B, are not the vertices of the focal axis, AP, BR meet the directrix (4. COR. 9.), and therefore the right lines touching any generating circle in the points reſpondent to A, B, will meet the directrix. Round C the centre of the ſections deſcribe the generating circle, paſſing through the vertices D, d, of the focal axis (4. COR. DEF.), and let Dd meet the directrix in I. Through the focus F draw Fab reſpondent to the diameter AB, which will meet the circle in a, b, reſpondent to the vertices AB (1. COR. 3.). Becauſe Dd, a diameter of the circle, is harmonically divided in F, and in I its concourſe with the directrix (3. COR. 1.), and through I one of the harmonic points is drawn the directrix at right angles to Dd, while through F the other harmonic point is drawn Fab, meeting the circle in the points a, b, the right line ab ſhall be harmonically divided in F and in its concourſe with the directrix (LEM. 10.), and therefore the right lines touching the circle in a, b, ſhall have their concourſe in the directrix (LEM. 11.). Wherefore the right lines AP, BR, touching the ſection in the reſpondent points A, B, are parallel between themſelves (4. COR. 11.).

Converſely. If two right lines AP, BR, parallel to each other, touch an ellipſe, on the oppoſite hyperbolas, in A, B; I ſay, that the points A, B, ſhall be diametrically oppoſite.

Let AB be joined, meeting the axis Dd in O, and round O deſcribe the generating circle, meeting the axis in E, e. Then, other things remaining the ſame, becauſe AP, BR, touching the ſection are parallel to each other, the tangents to the circle at the reſpondent points a, b, will meet in the directrix (4. COR. 11.). Let them meet in the directrix in the point G. Becauſe Ga, Gb, touch

touch the circle, and from their concourse G is drawn the right line GI, the diameter of the circle perpendicular to GI will be harmonically divided in the points I, F, in which it meets GI, AB (5. COR. LEM. 10.). Wherefore OE is a mean proportional between OI, OF (2. COR. LEM. 5.), and OI is to OF in the duplicate ratio of OI to OE. For the same reason, because Dd is harmonically divided in I, F (3. COR. 1.), and bisected in C, CI will be to CF in the duplicate ratio of CI to CD. But because O, C are the centres of the generating circles, whose semidiameters are OE, CD (4. COR. DEF.), OI will be to OE as CI is to CD (16. DEF.). Wherefore ex æquo OI is to OF as CI is to CF, and dividendo, FI is to OF as FI is to CF. OF is therefore equal to CF, viz. the points O, C are one and the same, and the points A, B are diametrically opposite. FIG.

COR. 1. The right lines touching an ellipse at the vertices of the axis minor, are parallel to the axis major, or perpendicular to the directrix.

For AB being then parallel to the directrix (12. DEF.), its respondent aFb will also be parallel to the directrix, viz. perpendicular to the axis Dd. Wherefore in this case, the tangents at a, b, which are the focal correspondents to AP, BR, will have their concourse in I (LEM. 9.). Join aI, CI. CI, aI therefore, pertaining to the circle, will be correspondent to AP, FP, pertaining to the section, and aI is parallel to FP, CI to AP, that is, AP is parallel to the axis minor, or perpendicular to the directrix. In the same manner it is proved, that BR is parallel to CI. 40.

COR. 2. The semi-axis minor of the ellipse, or semi-axis secundus of the hyperbola, is a mean proportional between the segments of the axis, intercepted between the vertices and focus, or between the segments intercepted between the focus and directrix, the focus and centre.

The same things remaining in the case of the ellipse, because aI is parallel to FP, and aF to IP, IaFP is a parallelogram, and 40.

aF

FIG. aF is equal to the opposite side IP. For the same reason, because AP is parallel to CI, and AC to HP, ACIP is a parallelogram, and AC is equal to IP. Therefore AC is equal to aF. But because of the circle, the square of aF is equal to the rectangle DFd, therefore the square of AC is equal to the rectangle DFd, or to the rectangle IFC (16. COR. DEF.), and AC is a mean proportional between DF, Fd, or between IF, FC.

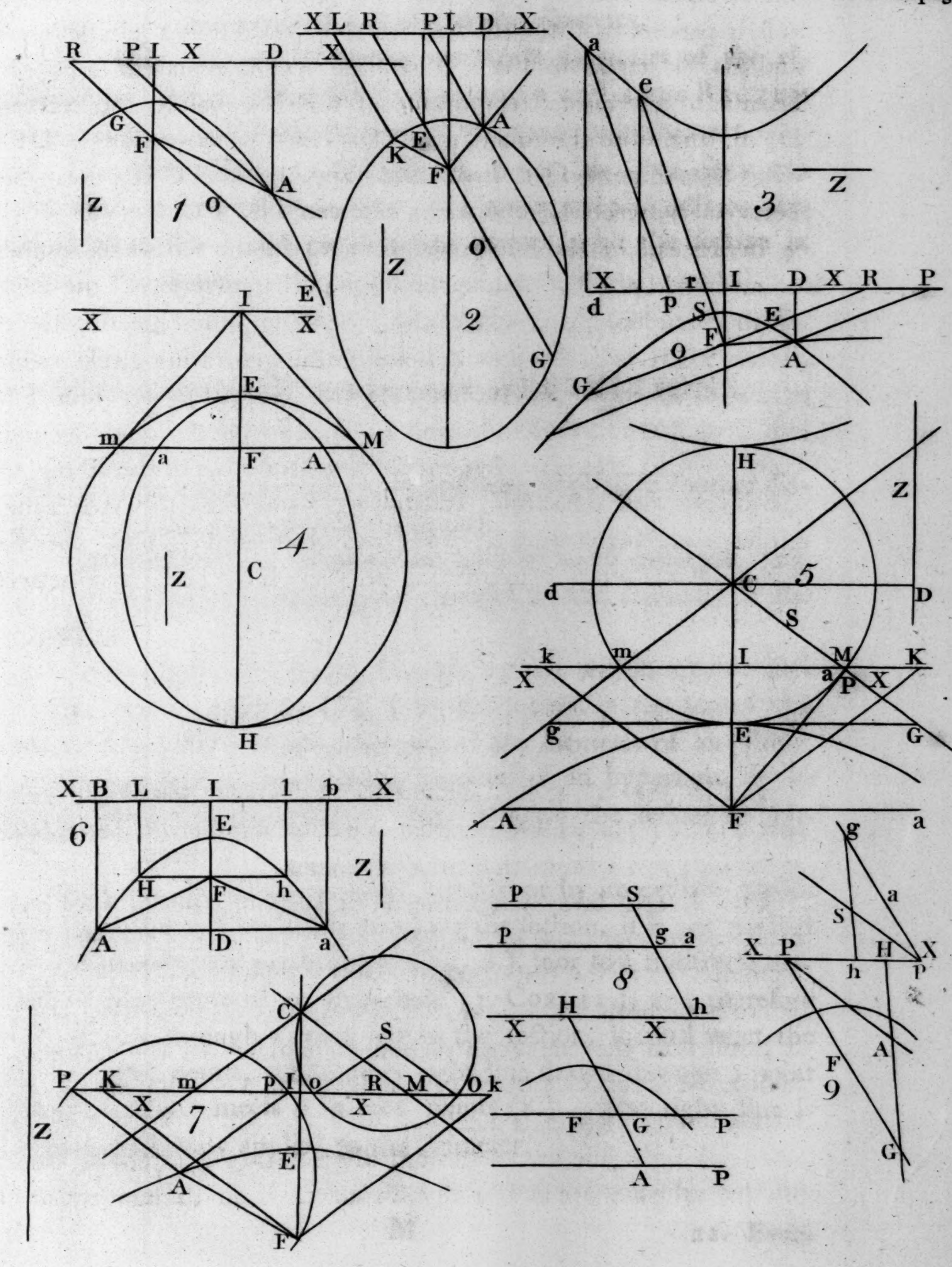
5. In the case of the hyperbola, let CD be the semi-axis secundus of an hyperbola, EC being the semi-axis transverse, C the centre of the hyperbola, and CM, Cm, the assymptotes meeting the directrix in M, m. Round C describe the generating circle, which will pass through E, M, m (4. & 8. COR. DEF.), join FM, which will touch the circle, and be equal to the semi-axis secundus cD (19. COR. DEF.). But for the same reason as in the ellipse, viz. the property of the circle, and the 17. COR. DEF., FM is a mean proportional between FE, FH, and between IF, FC, therefore also CD is a mean proportional between EF, FH, and between IF, FC.

DEFINITIONS.

22. In the ellipse and hyperbola, a diameter parallel to a right line touching the section in either vertex of another diameter, is said to be CONJUGATE to that other diameter.

23. A right line, which meeting a conic section in two points meets a diameter, and is parallel to a tangent at the vertex of that diameter, or to the conjugate diameter; or, which meeting the opposite hyperbolas, each in one point, meets a diameter secunda, and is parallel to the transverse diameter conjugate to the secunda, is said to be ORDINATELY APPLIED, or to be an ORDINATE to that diameter, which it meets.

24. A right line drawn through the focus of a parabola, parallel to the ordinates applied to any diameter, or to the tangent at the vertex



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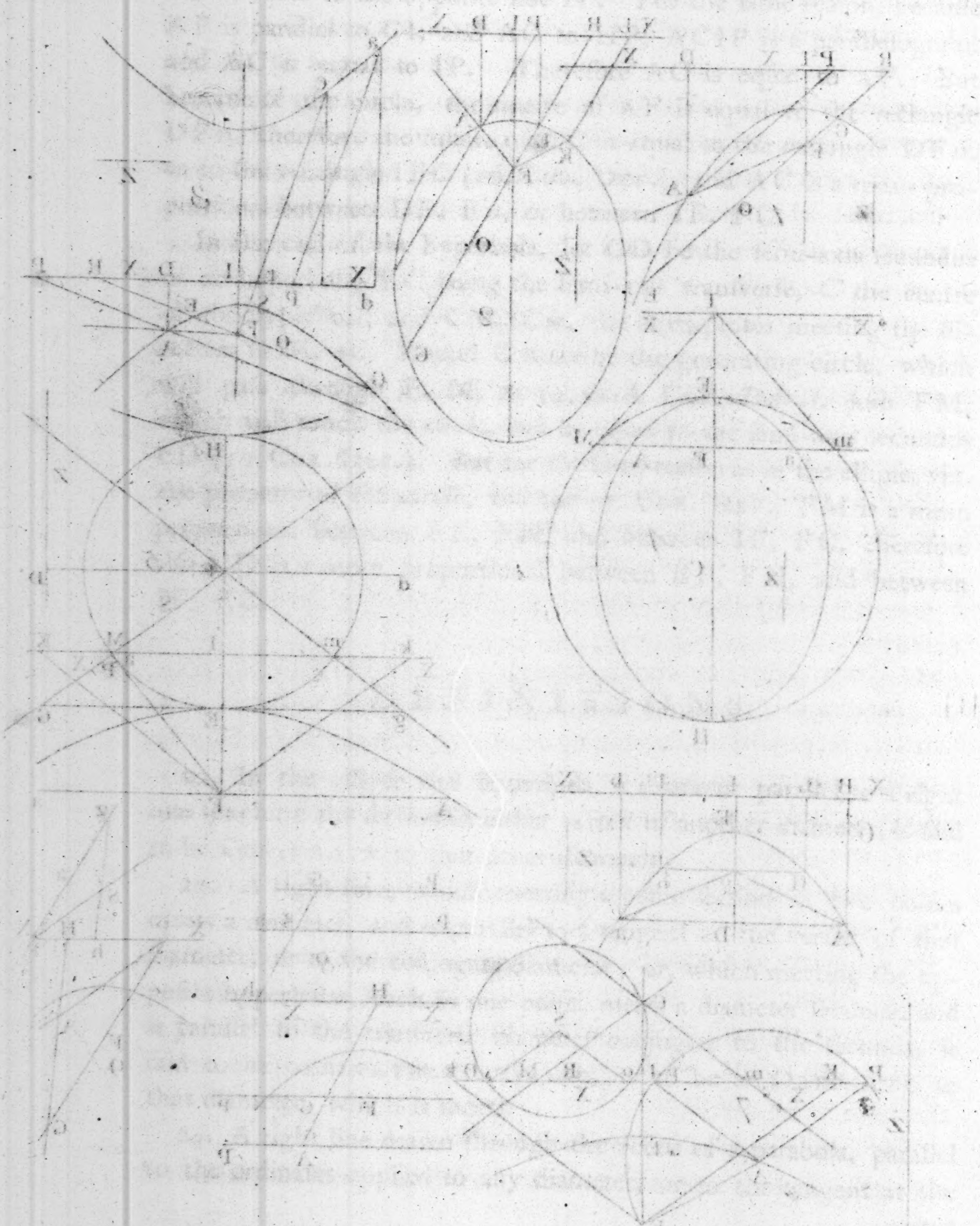


vertex of the diameter, and intercepted by the section, is called the PARAMETER, or LATUS RECTUM of the diameter. FIG.

25. A third proportional to two conjugate diameters of the ellipse or hyperbola, is called the PARAMETER OR LATUS RECTUM of that diameter, which is the first of the three proportionals.

26. If a circle touch a conic section, so that no other circle can be described which passes between this circle and the section, this circle is said to have the SAME CURVATURE with the section in the point of contact.

COROLLARIES FROM THE DEFINITIONS.

20. The diameter parallel to the ordinates applied to another diameter is conjugate to that other diameter.

For the ordinates to a diameter are parallel to its conjugate (23. DEF.), and therefore the diameter parallel to the ordinates is the conjugate.

21. Every right line drawn through a point within a conic section, parallel to a right line touching the section in the vertex of a diameter, or parallel to the conjugate of any diameter of an ellipse, or to the conjugate of a transverse diameter of an hyperbola, is ordinately applied to the diameter, viz. it meets the section or sections in two points.

Because the right line is either directly or by immediate consequence parallel to a right line touching the section, it is not parallel to the diameters of a parabola (1. COR. 6.), nor to a transverse diameter or assymptote of an hyperbola (3. COR. 11.), and therefore being drawn through a point within the section, it must meet the section in two points, while every right line drawn through a point within the ellipse meets it in two points (8.). The right line is therefore ordinately applied to the diameter.

FIG. 22. Every right line parallel to a transverse diameter of an hyperbola is ordinately applied to the diameter secunda, which is conjugate to the transverse.

For the right line meets each opposite section (7.), and therefore is ordinately applied to the conjugate (22. DEF.).

23. Every right line drawn through a point in a conic section, and not itself touching the section, but parallel to a right line touching the section, is ordinately applied to the diameter passing through the point of contact, viz. it meets the section again.

For the section lying wholly on the parts of the touching line towards the parallel, the parallel must meet the diameter within the section, and therefore must meet the section in another point (21. COR.).

24. The axes of the ellipse and hyperbola are conjugate.

For the axis secundus is parallel to the tangents at the vertices of the axis transverse, because each is parallel to the directrix (12. DEF. & 3. COR. 9.).

25. Two conjugate diameters of the ellipse and hyperbola are alternately proportional to their parameters.

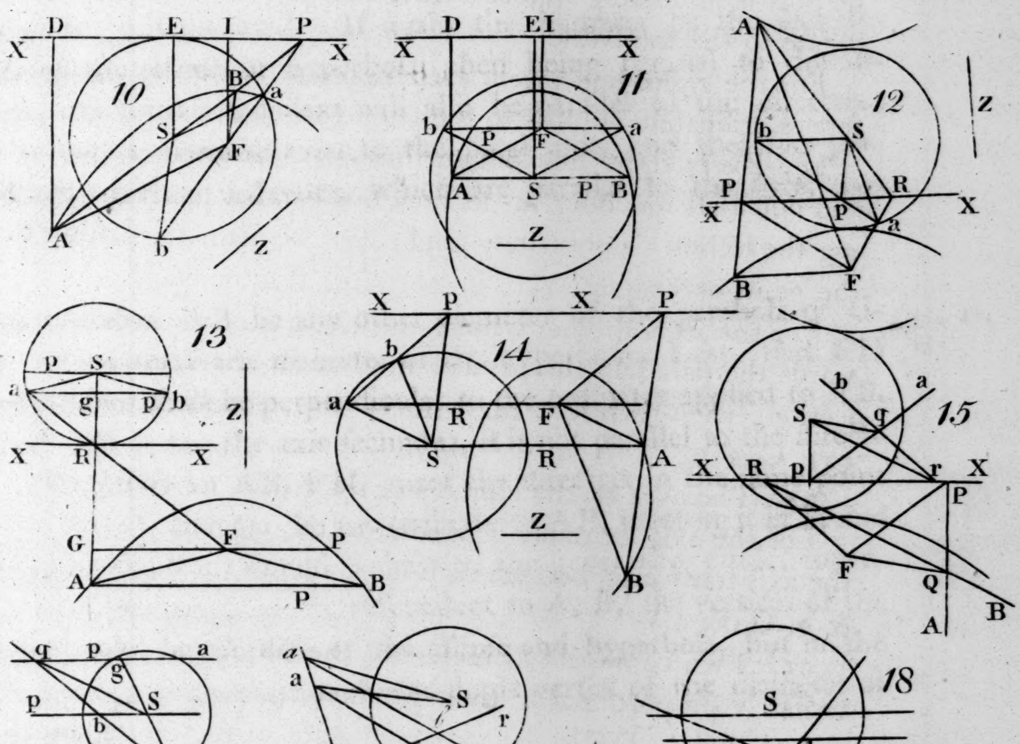
40. Let ACB, DCd, be two conjugate diameters of an ellipse or hyperbola, and L, l, represent their parameters. I say, that AB is to L as l is to Dd.

Because Dd is a mean proportional between AB, L, and AB a mean proportional between l, Dd; therefor AB is to L in the duplicate proportion of AB to Dd, and l is to Dd in the duplicate proportion of AB to Dd (10. DEF. e. 5.). Therefore ex æquo, AB is to L as l is to Dd.

PROP. 13.

The focal respondent to a diameter of a conic section is perpendicular to the ordinates applied to that diameter.

If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20.n 2.

If the diameter be the focal axis, the focal reſpondent coincides with this axis, and therefore is perpendicular to the ordinates applied to it (3. COR. 10.). If again the diameter be the axis ſecunda of the ellipſe or hyperbola, then being parallel to the directrix, the focal reſpondent will alſo be parallel to the directrix; viz. it will be perpendicular to the focal axis, and therefore perpendicular to thoſe ordinates, which are parallel to the focal axis (23. DEF.). FIG.

Let therefore AB be any other diameter of the parabola or ellipſe, or any tranſverſe diameter of the hyperbola; I ſay, that FH its reſpondent ſhall be perpendicular to the ordinates applied to AB. Becauſe AB is not the axis ſecundus, it is not parallel to the directrix. Wherefore let AB, FH, meet the directrix in the ſame point H (17. DEF.), and Qq be any ordinate to AB, meeting it in S, and the directrix in G. Round S deſcribe the generating circle, meeting FH in the points A, B, reſpondent to A, B, the vertices of the diameter AB, in the caſe of the ellipſe and hyperbola, but in the ſingle point A reſpondent to A the ſingle vertex of the diameter of the parabola. 41, 42, 43.

Becauſe the tangents to the ſection at the vertices of A, B, of the diameter AB are parallel between themſelves (12.), and Qq is drawn through the centre S of the generating circle, parallel to theſe tangents, therefore Qq pertaining to the circle, will be the conic correſpondent to each of them (18. DEF.), while GA, GB, drawn from the concourſe of Qq with the directrix to the reſpondent points A, B, will be the focal correſpondents to theſe tangents (3. COR. 3.), and they will alſo touch the circle in the points A, B (9.). For the ſame reaſon, in the parabola, GA will touch the circle in the ſingle point A reſpondent to the ſingle vertex A of its diameter, while the directrix GH alſo touches it in the point H, the concourſe of AB with the directrix (6. COR. DEF.). Wherefore in the ellipſe and hyperbola GA, GB, and in the para-

 bola,

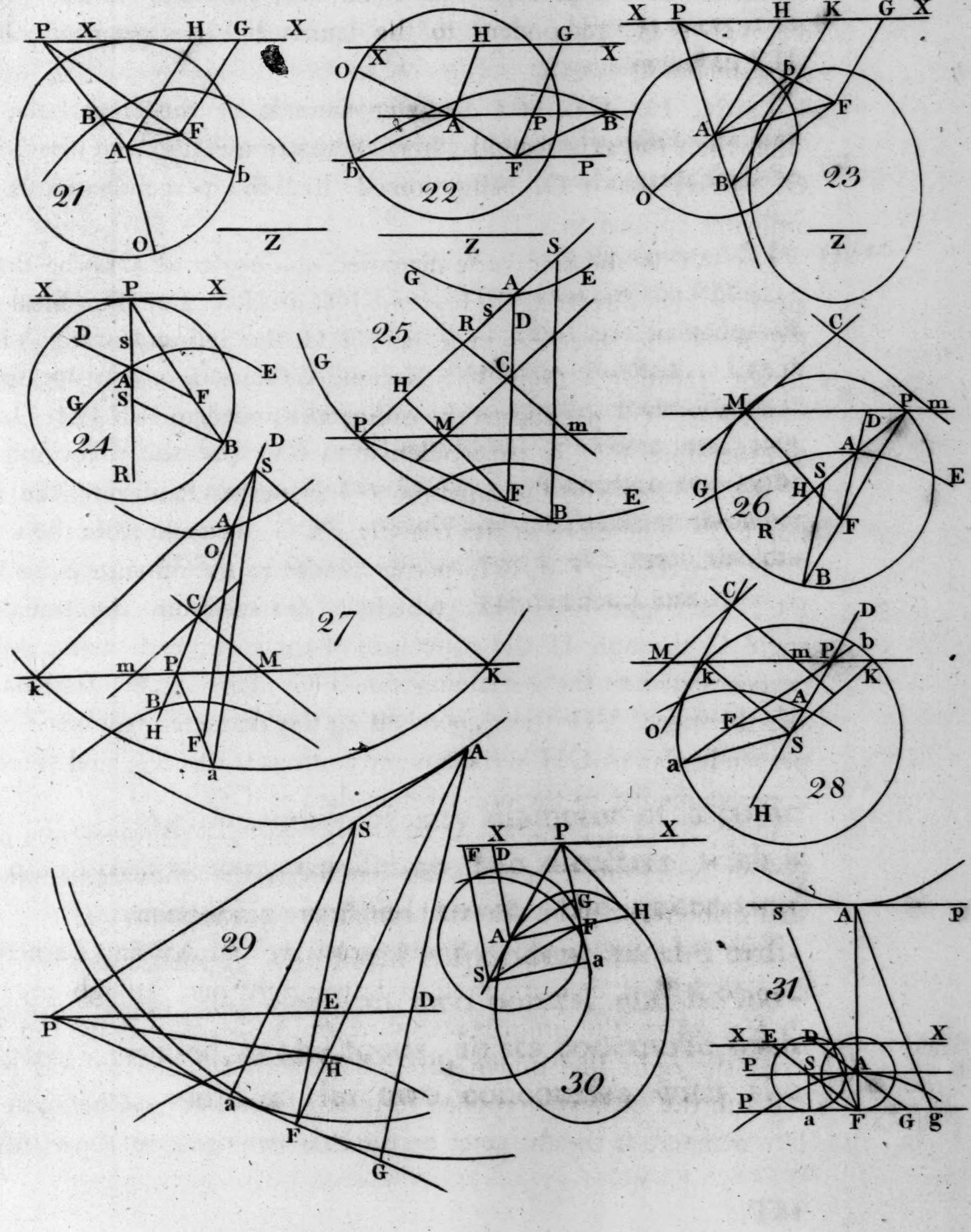
FIG. bola, GA, GH, touch the circle, and consequently SG which joins the centre S and the concourse of the tangents will be perpendicular to AB or FH which joins the points of contact (Lem. 8.), viz. FH, respondent to the diameter AB is perpendicular to its ordinate Qq.

38. Lastly, Let CG be a diameter secunda of the hyperbola, any other than the axis secunda, and therefore meeting the directrix in G. I say, that FG, being joined, shall be perpendicular to the ordinates applied to CG.

Let ACB the transverse diameter conjugate to CG be drawn, meeting the directrix in H, and FG in L. Let the focal axis FC meet the directrix in I, and FH the respondent to AB be drawn, meeting CG in V. Because CG is the conjugate diameter to AB, it is parallel to the ordinates applied to AB (23. Def.), and therefore FH is perpendicular to CG (by the preceding case of this). But also the directrix GI is perpendicular to the focal axis FC; wherefore in the triangle FCG, because from the angles F, G, are drawn FV, GI, perpendicular to the opposite sides CG, FC, the right line CH, which is drawn from the remaining angle C through H the concourse of these perpendiculars, will be perpendicular to the remaining side FG (Lem. 2.). In this case also therefore, FG the respondent to the diameter secunda CG is perpendicular to CH the diameter conjugate to CG, and therefore is perpendicular to the ordinates applied to CG.

Cor. 1. Hence the focus, directrix, and any diameter of a conic section being given, a right line parallel to the ordinates applied to that diameter may be drawn through any given point.

Let F be the focus, XX the directrix, and AB any diameter of a conic section, it is required to draw a right line through any point
35. E parallel to the ordinates applied to AB.—If AB be the focal axis, the right line drawn through E perpendicular to AB is parallel to the ordinates applied to it (23. Def. & 3. Cor. 9.). In like manner, if the diameter be the axis conjugate to the transverse one



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one of the ellipſe or hyperbola, the right line drawn through the point E parallel to the focal axis or perpendicular to the directrix is parallel to the ordinates applied to the conjugate axis. But if it be any other diameter, it will meet the directrix. Let it meet it in H, join FH, the right line drawn through E perpendicular to FH will be parallel to the ordinates applied to AB. FIG. 40. 42.

COR. 2. If a right line parallel to the ordinates applied to a diameter be perpendicular alſo to the diameter, the diameter ſhall be an axis of the ſection.

Let Qq parallel to the ordinates applied to a diameter AS be perpendicular to AS; I ſay that AS ſhall be an axis of the ſection. 41, 42.

For if AS be parallel to the directrix, it is the axis ſecundus of the ellipſe or hyperbola, and if not, it meets the directrix. Let it meet it in H. Then it either paſſes through the focus F, or it does not. If it does, the COR. is admitted; if it does not, join FH, which will be perpendicular to Qq. Therefore two right lines FH, AH, meeting in H, are each perpendicular to the ſame right line Qq, which is abſurd. Wherefore if AS meet the directrix, it paſſes through the focus, viz. it is the focal axis.

PROP. 14.

If round the concourſe of any diameter of a conic ſection which is not parallel to the directrix with a right line ordinately applied to it, the generating circle be deſcribed, the focal reſpondent to the ordinate ſhall meet the circle in two points, and be harmonically divided in the focus, in its concourſe with the directrix, and in its two concourſes with the circle.

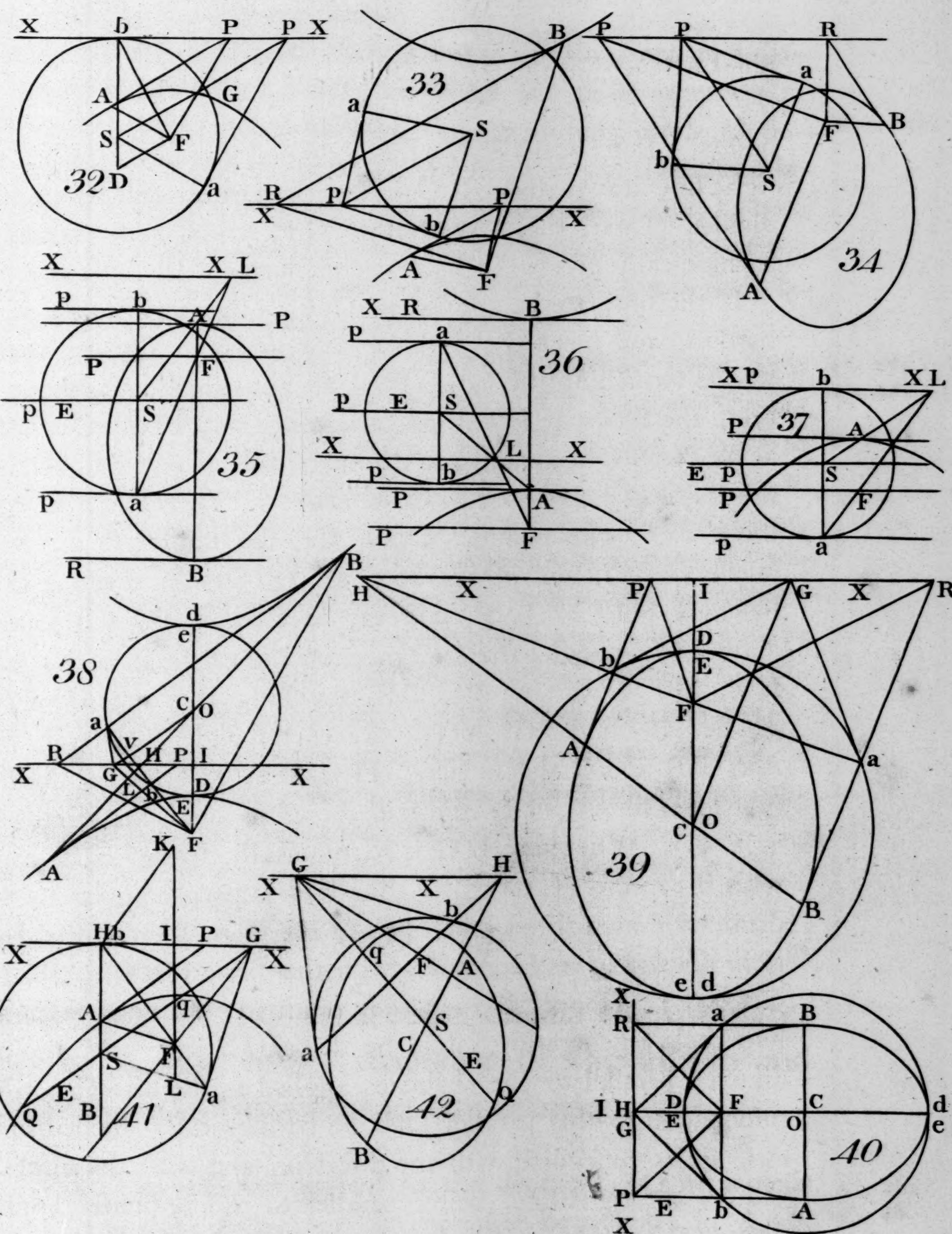
The

FIG. 41, 42, 43. The ſame things remaining, becauſe univerſally in the parabola and ellipſe, and alſo in the hyperbola, when the ordinate is applied to a tranſverſe diameter, viz. when it meets one ſection only, the point S is within the ſection, the point F will be within the generating circle (2. COR. 5.); therefore FG does meet the circle in two points. But, becauſe G A, G B, touch the circle in A, B, every right line drawn through G to cut the circle, will be harmonically divided (LEM. 9.). Wherefore G F which is ſo drawn will be harmonically divided in its two concourſes with the circle.

44. If the ordinate be applied to CH a diameter ſecunda of the hyperbola, and meet the oppoſite hyperbolas in Q, Q; then, other things remaining the ſame, becauſe the point S is without the ſection, the focus F is without the circle (2. COR. 5.). But QQ, being parallel to a tranſverſe diameter, will meet the directrix within the circle (15. DEF. & 8. COR. DEF.). Therefore FG does meet the circle in two points. Draw CP the tranſverſe diameter conjugate to CH, and parallel to QQ (23. DEF.), meeting the oppoſite hyperbolas in A, B, and the directrix in P; alſo joining FP, draw A G B parallel thereto, meeting the circle in A, B; laſtly join H A, H B, and draw A R, BR, touching the oppoſite ſections in A, B. Becauſe SG, A G B, are correſpondent to AB, FP, and A R, BR touch the ſections, H A, H B, will be the focal correſpondents to A R, BR, and will therefore touch the circle (3. COR. 3. & 9.). Wherefore HG is harmonically divided in its concourſes with the circle (LEM. 9.). But FH is perpendicular to SG, the diameter drawn through G (13.), therefore FG will be harmonically divided in its concourſes with the circle (LEM. 10.).

COR. 1. Hence, if round any point in a diameter ſecunda of an hyperbola a generating circle be deſcribed, and from the point be drawn a parallel to the ordinates applied to the diameter, the focal reſpondent to this parallel ſhall be harmonically divided in the focus, in its concourſe with the directrix, and with the circle.

For, the right line, being parallel to the ordinates applied to a diameter ſecunda, is parallel to a tranſverſe diameter (22. & 23. DEF.),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DEF.), and therefore will meet each of the opposite hyperbolas FIG.
(7.). Wherefore the parallel is ordinately applied to the diameter, and the corollary is the PROP. itself.

COR. 2. If round any point within a conic section, and in a diameter which is not the focal axis, or round any point whatever in a diameter secunda of the hyperbolas, the generating circle be described, and a right line drawn from the focus to cut this circle be harmonically divided in the focus, in its concourses with the circle and with the directrix; the right line drawn from the centre of the circle to the concourse of this focal line with the directrix shall be parallel to the ordinates applied to the diameter.

The same things remaining, if a right line drawn from F to meet the directrix in G, and cut the circle, be harmonically divided in its concourses with the circle, and SG be not parallel to the ordinates applied to the diameter AB in the first case, or to SC in the second; then, because the diameter is not the focal axis, an ordinate applied through S to the diameter will meet the directrix, and in some other point than G; and a right line drawn through this other point will meet the circle in two points, and be harmonically divided in them. But this is absurd, because the diameter SF not being the focal axis, is not perpendicular to the directrix; and therefore only one right line can be drawn from F to cut the circle and directrix, and be harmonically divided in its concourses with them (4. COR. LEM. 10.). 41, 42, 43, 44.

PROP. 15.

A diameter of a conic section bisects the ordinates applied to it, and every diameter of the ellipse and hyperbola is itself bisected in the centre.

Let AB be an ordinate applied to the diameter SO of a conic section, meeting the section or the opposite hyperbolas in A, B, and 45, 46, 47, 48.

FIG. and the diameter in S; I say, that AB shall be bisected in S.—
49. And if AB be a diameter of an ellipse or hyperbola, viz. an ordinate which passes through the centre; I say, that AB is bisected in the centre.

CASE 1. When the diameter is the focal axis.

45. Round S describe the generating circle, and, F being the focus, XX the the directrix of the section, draw FP respondent to AB, meeting the circle in A, B, respondent to the points A, B in the section. Because AB, being ordinately applied to the focal axis, is parallel to the directrix, FA is equal to SA, and FB to SB (6. COR. 3.). But because the axis FSO of the section is perpendicular to AB, and therefore also to AB, it will bisect AB in F (3. e. 3.); wherefore FA being equal to FB, AS will be equal to BS.

CASE 2. When the ordinate is applied to any other diameter.

46, 47, 48, 49. Every thing in this case remains the same, except that AB is not parallel to the directrix, but meets it in a point P, through which the focal respondent FP also passes. As before, AF is parallel to AS, and BF to BS (3.), and also FP is harmonically divided in A, B (14.). Therefore PA is to AF as PB is to BF, and on account of the parallels, PA is to AF as PS is to AS, and PB is to BF as PS is to BS. Wherefore ex æquo, PS is to AS as PS is to BS, and AS is equal to BS.

Otherwise. Round S describe the generating circle, and F being the focus, XX the directrix, draw FP respondent to AB, meeting the circle in the points A, B, respondent to A, B, in the section. Join FA, FB, SA, SB, FS, and draw FI parallel to SP. If SO be the focal axis, it passes through F, and AB being ordinately applied to it, is perpendicular to SO, and parallel to the directrix. Wherefore FP is also parallel to the directrix, viz. it is perpendicular to SF, and consequently AB is bisected in F. (3. e. 3.). In the triangle ASB therefore, the right line SF being drawn from

from the vertex S to bisect the base, and SP parallel to the base, the four right lines SP, Sa, SF, Sb, will be harmonicals (3. Cor. Lem. 4.). If SO be any other diameter, AB, FP, will meet the directrix in the common point P (1. Cor. 6.), and because in this case, ab is harmonically divided in P, F (14.), SP, Sa, SF, Sb will, as before, be harmonicals. But FA is parallel to Sa, FB to Sb (3. & 19. Def.), and FI is parallel to SP, therefore FA, FS, FB, FI are also harmonicals (2. Cor. Lem. 4.), and consequently AB parallel to one of them FI, and meeting the other three in A, S, B, will be bisected in the middle concourse S (Lem. 4.). FIG.

I say also, that every diameter of the ellipse and hyperbola is bisected in the centre. For every diameter of the ellipse, and every transverse diameter of the hyperbola is an ordinate applied to its conjugate, and therefore this is only a case of the general property which has been demonstrated of an ordinate, and the demonstration as applied to this case is in all respects the same. Vide Fig. 48, 49. While every diameter secunda of the hyperbola is, from its very definition, bisected in the centre. Vide Note 15. Def. 48, 49.

Cor. 1. Every right line terminated by a conic section, or the opposite hyperbolas, and bisected by a diameter, is ordinately applied to that diameter.

For if it be not ordinately applied to this diameter, then this diameter is not conjugate to the diameter parallel to the right line, and this conjugate, if it were drawn, would also bisect the right line; that is, two different right lines would pass through the same two points, the centre and the point of bisection; or in the case of the parabola, two parallel right lines would pass through one and the same point of bisection, which is absurd.

Cor. 2. If two or more parallel right lines be terminated by the same section, or by the opposite hyperbolas, the diameter which bisects one of them, bisects them all.

For as that which is bisected is ordinately applied to the diameter, they are all ordinately applied thereto, and therefore, &c.

Cor. 3. A right line, which bisects two or more parallels terminated by a conic section, or by the opposite hyperbolas, is a

FIG. diameter, and that diameter to which the parallels are ordinately applied.

For the diameter which biſects one biſects all the parallels; and if this diameter be any other than the right line already biſecting them, then would two different right lines be drawn through the ſame two points or more, which is abſurd. Wherefore the biſecting line is the diameter itſelf, viz. that to which the parallels are ordinately applied (2. Cor.)

Cor. 4. For the ſame reaſon, if there be two parallels, one of which touches a conic ſection, but the other meets it in two points, the right line which paſſes through the point of contact in the one, and the point of biſection in the other, is a diameter, and that diameter, to whoſe ordinates both the right lines are parallel.

Cor. 5. A diameter, biſecting one or more parallels to another diameter, is conjugate to that other diameter.

For the parallels are ordinately applied to the diameter biſecting them (2. Cor.), and therefore the diameter to which they are parallel, will be conjugate to the diameter biſecting them (23. Def.).

Cor. 6. Two right lines terminated by a conic ſection, or by the oppoſite hyperbolas, and which do not both paſs through the centre, cannot mutually biſect each other.

For, if it were poſſible, then would each of the right lines be ordinately applied to the diameter paſſing through the concourſe, viz. they would each be parallel to the conjugate diameter, which is abſurd.

Cor. 7. If an ordinate be applied to the focal axis, and round its concourſe with the axis the generating circle be deſcribed, the focal reſpondent to the ordinate, intercepted by the circle, ſhall be equal to the ordinate intercepted by the ſection.—This is manifeſt,
45. becauſe SAFa, SBFb are parallelograms.

Cor. 8. If a right line parallel to the ordinates applied to a diameter be biſected in its concourſe with the diameter, and one of its extremities be in the ſection, the other ſhall be in the ſection alſo.

Let

Let Qq ordinately applied to a diameter SC, meet it in S, and be bisected in S. If Q be in the section, the point q shall be in the section also.—Because Qq parallel to the ordinates meets the section in Q, it will fall within the section, if SC be not a diameter secunda of the hyperbola, and therefore meet the section again (8.), or if SC be a diameter secunda, Qq will meet the opposite section (7.). If not in q therefore, let it meet the section or the opposite section in r. Then Sr is equal (to SQ, viz.) to Sq, the part to the whole, which is absurd. Wherefore QS meeting again the section, or the opposite section, can meet it in no other point than q, viz. q is in the section. FIG. 43, 44.

COR. 9. If two opposite sides of a quadrilateral inscribed in a conic section, or in the opposite hyperbolas, be parallel; and a third parallel be drawn, either touching the same section or either opposite hyperbola, or meeting the same or the opposite sections in two points, the segments of the third parallel intercepted between the section and the two remaining sides of the quadrilateral shall be equal between themselves.

Let ABCD be a quadrilateral so inscribed, having two opposite sides AD, BC, parallel between themselves, and let EF be a third parallel to AD or BC, either touching the same, or an opposite section in L, or meeting the section or sections in O, P, and meeting AB, CD, the remaining sides of the quadrilateral in E, F; I say, that the intercepted segments EL, FL, in the one case, and EO, FP, in the other, shall be equal between themselves. 50, 51.

Bisect AD, BC, in H, I, join HI meeting EF, in the second case, in L. Because, in both cases, HI is the diameter to which AD, BC, are ordinately applied (3. COR.), therefore, in the first case, EF being parallel to the ordinates AD, BC, and touching the section, must touch it in a vertex of the diameter HI (23. DEF.), that is, HI passes also through the point of contact L. In both cases then, because the three right lines AB, HI, CD, fall upon the parallels AD, BC, and AD, BC, being each bisected in H, I, are proportionally divided by them, therefore every right

 line

FIG. line parallel to AD, BC, and meeting AB, HI, CD, will be divided in the same proportion (LEM. 1.), viz. will be bisected by HI. Therefore EL is equal to FL, and, in the second case, because OL is equal to PL, the remainder EO is equal to the remainder FP.

COR. 10. If a triangle be inscribed in a conic section, or in the opposite hyperbolas, and a right line parallel to one of the sides, meet the section in two points, or each of the opposite hyperbolas, the segments intercepted between the section and the other two sides of the triangle, shall be equal between themselves.

This is a case of the preceding Corollary, when two adjoining angles of the quadrilateral coalesce, and become one; and the separate demonstration of it is similar to, but much simpler than the preceding.

PROP. 16.

If a right line meet a conic section in two points, or each of the opposite hyperbolas in one point, the respondent focal line shall make equal angles with the right lines drawn from the focus to the points of concourse with the section; externally, if the right line meet the same section; but internally, if it meet the opposite hyperbolas.

45, 46. Every thing remaining the same as in the last, because AF is
47, 48. parallel to Sa, and BF to Sb, the angle AFb is equal to the
49. angle Sab, and the angle BFa to the angle Sba. But from the property of the circle, the triangle Sab, being isosceles, the angle Sab is equal to the angle Sba; therefore the angle AFb is equal to the angle BFa. And if AB meet the same section, the respondent line FP is either parallel to AB, or meets it in the directrix without

without the terms A, B; therefore in this case, FP falls entirely without the triangle AFB. But when AB meets the opposite hyperbolas, it will meet the directrix between the terms A, B, and therefore FP, which is drawn to the same point in the directrix, will fall within the triangle AFB. Wherefore, in the one case, FP makes equal angles, externally, in the other, internally, with AF, BF. FIG.

PROP. 17.

If a right line ordinately applied to a diameter of a conic section meet the same section in two points, or the opposite hyperbolas, each in one point, and round the concourse of the ordinate with the diameter the generating circle be drawn; the right lines drawn from the focus to the vertices of the ordinate shall together be equal to the diameter of the circle, if the ordinate meet the same section; but if it meet the opposite hyperbolas, then the excess by which the greater of these right lines exceeds the less, shall be equal to the diameter of the circle.

If the ordinate pass through the focus, the diameter of the generating circle is the ordinate itself (2. COR. DEF.), and this ordinate is equal to the right lines drawn from the focus to the vertices, taken together, or to their excess, accordingly as the ordinate meets the same section, or the opposite hyperbolas, therefore 45, 46.
in this case the PROP. is already demonstrated. But if the ordinate 47, 48.
do not pass through the focus, then every thing remaining the same 49.
as in the last PROP., join Ab, Ba, and let BF meet Ab in E, also draw AG parallel to ab, meeting BF in G. Then, because of

FIG. of the parallels the angle GAF is equal to the angle AFb, and the angle AGF is equal to the angle BFa; but the angle AFb is equal to the angle BFa (16.), therefore the angle GAF is equal to the angle AGF, and AF is equal to FG (6. e. 1.). Again, because Sb is parallel to BF or BE, and AS is equal to BS (15.), Ab will be equal to Eb (2. e. 6.), therefore for the same reason will GF be equal to FE. Wherefore FE is equal to AF, and BF being common, the whole or remainder BE will be equal to AF, BF together, or to the excess of AF, BF. But Sb being parallel to BE, and AB double to SA, BE will be double to Sb. Therefore AF, BF together, or the excess of AF, BF is double to Sb the semidiameter of the generating circle, and consequently equal to the whole diameter.

Cor. 1. If an ordinate to any diameter of a conic section pass through the focus, and round its concourse with the diameter the generating circle be drawn, this circle will pass through the vertices of the ordinate, viz. the semidiameter of the circle will be the semi-ordinate itself.

Cor. 2. If round the centre of the ellipse or hyperbola, or the concourse of an ordinate with the diameter to which it is applied, the generating circle be described, and the focal respondent to a diameter of the section or to the ordinate be drawn, meeting the circle in two points respondent to the vertices of the diameter or ordinate, the right line joining either vertex and the alternate respondent point in the circle shall be perpendicular to the focal respondent; and in the case of the diameter, the right line so drawn shall touch the section in the vertex.

Because AF, GF, EF, are equal between themselves, the angle GAE is an angle in a semicircle described round the centre F, and therefore a right angle (31. e. 3.), viz. AE or Ab is perpendicular
48, 49. to AG, and consequently to its parallel ab. When AB is a diameter of the section, and A one of its vertices, because AE is perpendicular to ab the focal respondent to AB, it will be parallel to

to the ordinates applied to AB (13.), and therefore touch the section in A (23. DEF.). FIG.

COR. 3. Hence, if through the focus of an ellipse or hyperbola a perpendicular be drawn to the tangents at the vertices of any diameter, each concourse with the tangents will be in the circumference of the generating circle described round the centre of the section. Or, what is the same thing, the right line drawn from the centre of the sections to either concourse, will be equal to the semi-axis transverse of the section.

COR. 4. In the ellipse, the right lines drawn from the focus to the vertices of any diameter are together; but in the hyperbola, the excess of these lines is equal to the axis transverse.

This is the PROP. itself, when the ordinate is a diameter of the section, and the semidiameter of the generating circle is the semi-axis transverse (4. COR. DEF.).

COR. 5. The latus rectum of a diameter of a parabola is double to the segment of the diameter intercepted between the directrix and its focal ordinate, and is quadruple to the segment intercepted between the directrix and vertex.

When the diameter is the axis FX. Let it meet the directrix in X, the section in G, and AFB be its focal ordinate. By this PROP. AB is double to the semidiameter of the generating circle described round F, viz. to FX (6. COR. DEF.); and FX is double to GX (7. COR. 1.). Therefore AB, which is the parameter of the axis (5. COR. 1.), is double to FX, and quadruple to GX.—If the focal ordinate DFE be applied to any other diameter QI, which meets the section in H, the directrix in I, and the ordinate in Q. Then for the same reason ED is double to QI, and QI is double to HI (8. COR. 9.), therefore ED, which is the parameter of the diameter QH (24. DEF.), is double to QI, and quadruple to HI. 54.

PROP.

FIG.

PROP. 18.

If two right lines touching a conic ſection, or the oppoſite hyperbolas, meet each other, the right line joining the points of contact ſhall be ordinately applied to the diameter drawn through the concourſe of the tangents.

78, 79. Let AS, BS, touching a conic ſection, or the oppoſite hyper-
88. bolas, in A, B, meet in S, and AB be joined; I ſay, that AB is ordinately applied to the diameter drawn through S.

Round S deſcribe the generating circle, and draw FA, FB, reſpondent to AS, BS, meeting the circle and touching the circle in the points A, B, reſpondent to A, B (9.). Then,

88. CASE 1. When AB is parallel to the directrix.

Join FS. Becauſe FA, FB touch the circle, FS which is drawn from the centre S to the concourſe of the tangents will be perpendicular to AB (LEM. 8.). But AB is the focal correſpondent to AB (2. COR. 3.), and therefore parallel alſo to the directrix (18. DEF.) Wherefore FS will be perpendicular to the directrix, viz. it will be the axis of the ſection, and AB being parallel to the directrix, will be ordinately applied to it.

78. CASE 2. When AB meets the directrix, and the right lines touch the ſame ſection.

Let AB meet the directrix in P, and join FP, the focal reſpondent to AB. Therefore AB, which is the focal correſpondent to FP, will alſo meet the directrix in a point P, and SP, being joined, will be the conic correſpondent (4. COR. 3.). Draw SR, the diameter of the ſection, which paſſes through S, and FR its reſpondent.

ſpondent. I ſay, that AB is ordinately applied to SR. Becauſe AS, BS, touch the ſame ſection, SR meets the ſection in two points, its vertices; therefore FR, its focal reſpondent, will meet the circle in two points D, E, reſpondent to theſe vertices (1. COR. 3.), and the tangents to the circle at theſe reſpondent points D, E, will have their concourſe in the directrix (4. COR. 11.). And becauſe AF, BF, touch the circle in A, B, and from their concourſe F is drawn FR to cut the circle in D, E, the right lines touching the circle in D, E, will have their concourſe in AB (3. COR. LEM. 10.). The tangents therefore at D, E, will meet in the point P, and conſequently SP will be perpendicular to DE or FR. But SP is parallel to AB (18. DEF.), therefore AB is perpendicular to FR, the focal reſpondent to the diameter SR, and conſequently is ordinately applied to the diameter SR (13.) FIG.

CASE 3. When the right lines touch the oppoſite hyperbolas. 79.

Here every thing remains the ſame, except that SR in this caſe becomes a diameter ſecunda, and therefore not meeting the oppoſite ſections, its focal reſpondent FR will not meet the circle. But in this caſe, becauſe AB is parallel to a tranſverſe diameter, SP, which is correſpondent and parallel to AB, will meet the directrix in a point P within the circle (15. DEF.); and therefore, becauſe FA, FB, touch the circle, FP, being drawn, would be harmonically divided in its two concourſes with the circle (LEM. 9.). Wherefore SP is parallel to the ordinates applied to the diameter SR (2. COR. 14.), and conſequently AB, being parallel to SP, is ordinately applied to SR.

COR. 1. If two right lines, touching a conic ſection or the oppoſite hyperbolas, meet each other, the right line drawn through the concourſe, and biſecting the right line which joins the points of contact, will be a diameter of the ſection.

For if not, yet the diameter drawn through the concourſe will biſect the ſame right line, viz. two different right lines may be drawn through the ſame two points, which is abſurd.

FIG. Cor. 2. For the ſame reaſon, the diameter drawn through the point of biſection, will paſs through the point of concourſe.

Cor. 3. If a right line drawn through the focus, be ordinately applied to the focal axis, the right lines drawn from the concourſe of the axis with the directrix to the points in which the focal ordinate meets the ſection ſhall touch the ſection.

Or, the right lines touching the ſection in the vertices of the focal ordinate, ſhall meet in the concourſe of the axis with the directrix.

For, by this Prop., the right lines touching the ſection in theſe points, do meet in the axis, and they alſo meet in the directrix (5. Cor. 9.); therefore they muſt meet in the concourſe of the axis with the directrix.

Cor. 4. If two right lines touching a conic ſection or the oppoſite hyperbolas, meet each other, and from one of the points of contact be drawn a parallel to the diameter paſſing through the concourſe of the tangents, the right line touching the ſection in the other point of contact, and intercepted between the contact and the parallel, ſhall be biſected in the concourſe of the tangents.

87, 89. The ſame things remaining as in the propoſition, from either point of contact A draw AH parallel to the diameter SR, meeting BS in H; I ſay, that BH is biſected in S.

Let SR meet AB in O. Becauſe AB is biſected in O, BH will be biſected in S (2. e. 6.).

87. Schol. In the parabola, AH is itſelf a diameter, but in the
89. hyperbola and ellipſe, if the diameter BCI be drawn, AH will paſs through the other vertex. Join AI, then becauſe AB is biſected in O, and BI in C, AI will be parallel to OC or SR, viz. AH and AI will be one and the ſame right line.

PROP.

PROP. 19.

If two right lines touch a conic section, the right line drawn from the focus to the concourse of the tangents, if they be not parallel; but, if they be parallel, the right line drawn through the focus parallel to the tangents, shall make equal angles with the right lines drawn from the focus to the points of contact; internally, if the right lines touch the same section, but externally, if they touch the opposite hyperbolas.

The same things remaining as in the last, join FS, FA, FB; 78, 79.
I say, that FS shall make equal angles with FA, FB, internally or externally, as the points of contact A, B, are in the same section, or in the opposite hyperbolas.

Because Fa, Fb, touch the circle, FS drawn from the concourse F to the centre S will be perpendicular to ab, viz. to FP, which is parallel to ab (18. Def.). But FP makes equal angles with AF, BF, externally or internally, as the points A, B, are in the same or opposite sections (16.) Wherefore FS, being perpendicular to FP, will make equal angles with AF, BF, internally or externally, as the points A, B, are in the same section, or in the opposite hyperbolas.

If the tangents Ab, Ba, be parallel to each other, draw FQ 48, 49.
parallel to Ab, Ba; I say, that FQ shall make equal angles with AF, BF, internally or externally, as above.

For FP is perpendicular to the ordinates applied to AB, which in this case becomes a diameter of the section (13. & 12.) viz. it is perpendicular to FQ. Therefore, for the same reason as above, FQ makes equal angles with AF, BF, internally or externally, as A, B, are in the same section, or in opposite hyperbolas.

FIG. Cor. Hence it appears, that the focal ordinate of any diameter of the ellipse or hyperbola makes equal angles with the right lines drawn from the focus to the vertices of the diameter, internally in the case of the ellipse, but externally in that of the hyperbola.

This is merely the latter case of the Prop.

PROP. 20.

If an ordinate to a diameter of a conic section pass through the focus, the rectangle under its segments between its vertices and the focus shall be equal to the rectangle under the semi-ordinate and the semi-latus rectum of the focal axis.

52, 53. Let AB, drawn through the focus F of a conic section, and
54. ordinately applied to a diameter meet the same in S, and the section in A, B; also let Z be the semi-parameter of the focal axis; I say, that the rectangles under AF, FB, and AS, Z, are equal between themselves.

54. If AB be parallel to the directrix, the diameter is the focal axis itself, in which case S coincides with F, and AS, AF, FB, become each equal to each other, and each equal to Z (5. Cor. 1.); therefore the rectangle AFB is equal to the rectangle under AS, Z.

52, 53. If AB be ordinately applied to any other diameter, and therefore meet the directrix in P, the generating circle described round S will pass through A, B (2. Cor. Def.), wherefore SA is to SP as Z is to FP (1. Cor. Def.). But AB is harmonically divided in F, P, (3. Cor. 1.), and it is bisected in S, therefore FS is to AS as AS is to SP (2. Cor. Lem. 5.), and ex æquo, FS is to AS as Z is to FP. The rectangle SFP is therefore equal to the rectangle under AS, Z. But because of the harmonic section, the rectangle AFB is equal to the rectangle SFP. Therefore the rectangle AFB is equal to the rectangle under AS, Z.

Cor.

FIG.

COR. If two right lines meeting each other in the focus, do each meet the ſame ſection in two points, or one the ſame, but the other the oppoſite hyperbolas, or both meet the oppoſite hyperbolas; the rectangles under the ſegments of each between the focus and ſection or ſections, ſhall be to each other as the ſegments of the focal lines, intercepted between the ſection or ſections.

The ſame things remaining, let DE alſo paſſing through F meet the ſection or ſections in D, E; I ſay, that the rectangle AFB ſhall be to the rectangle DFE as AB is to DE.

Biſect DE in Q. The rectangle AFB is equal to the rectangle under AS, Z, and the rectangle DFE is equal to the rectangle under DQ and Z. Therefore the rectangle AFB is to the rectangle DFE (as the rectangle under AS, Z, is to the rectangle under DQ, Z, viz. as AS is to DQ (1. e. 6.), viz. doubling the conſequents,) as AB is to DE.

PROP. 21.

If a right line touch a conic ſection, or meet the ſame ſection in two points, or each of the oppoſite hyperbolas, and round any point in the right line the generating circle be deſcribed; the ſquare of the ſegment intercepted between the centre of the circle and the point of contact, or the rectangle under the ſegments intercepted between the centre and the ſection or ſections, ſhall be to the ſquare of a right line drawn from the focus to touch the circle, or to the rectangle under the ſegments of a right line drawn from the focus to cut the circle, in the ratio of equality, if the right line be parallel to the directrix; but

if

FIG. if it meet the directrix, as the ſquare of its ſegment intercepted between the centre and the directrix is to the ſquare of a right line drawn from the concourſe with the directrix to touch the circle, or to the rectangle under the ſegments of a right line drawn from the concourſe with the directrix to cut the circle.

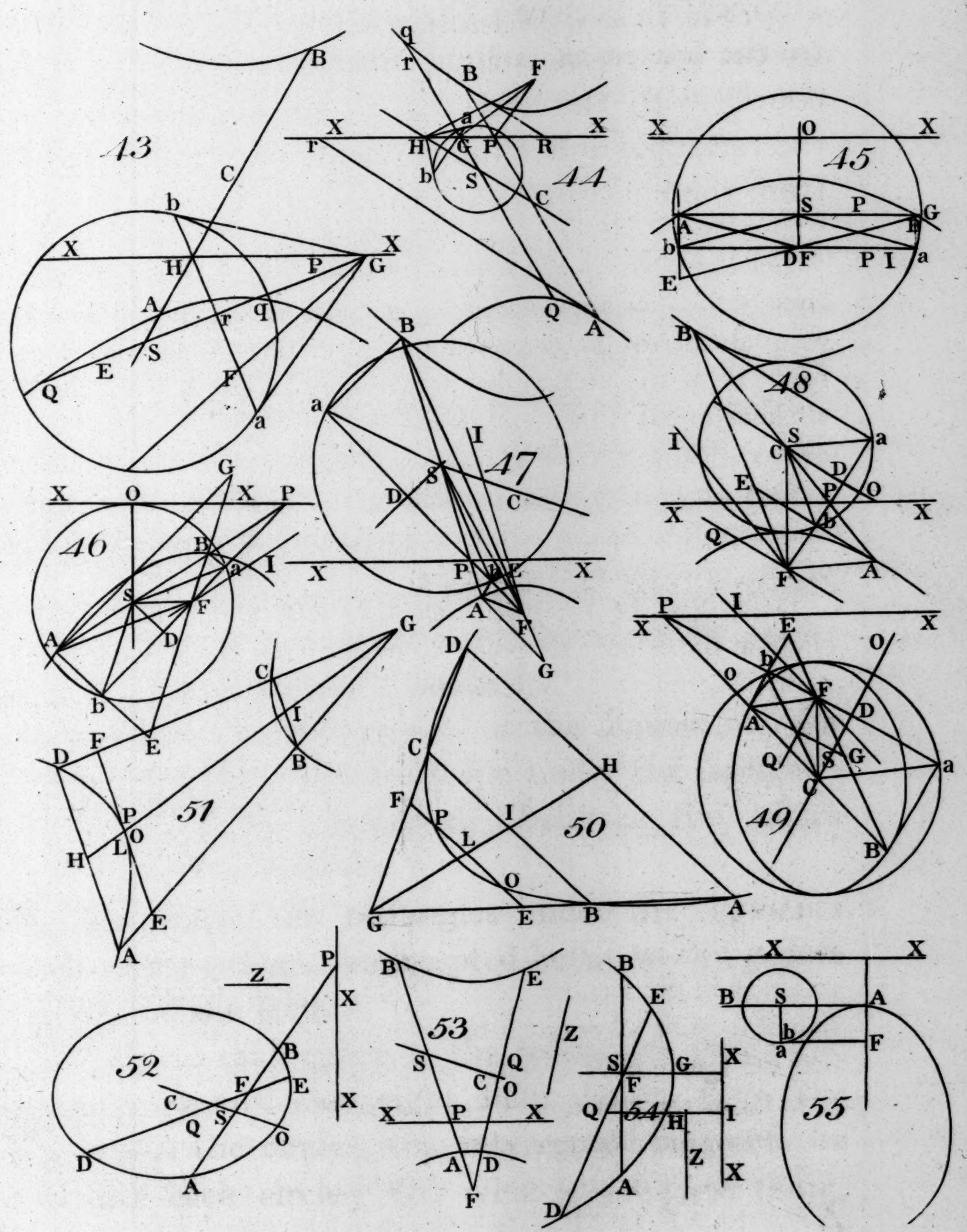
55, 56. 57, 58. Let the right line AB touch a conic ſection in A, or meet the ſame ſection or the oppoſite hyperbolas in A, B, and round any point S therein be deſcribed the generating circle, and from the focus F of the ſection be drawn a right line Fa to touch the circle or to cut it in a, b, then,

55, 56. Case 1. If AB be parallel to the directrix, I ſay, that the ſquare of SA, or the rectangle ASB, ſhall be equal to the ſquare of Fa, or to the rectangle aFb.

Draw Fa the focal reſpondent to AB, which will touch the circle in the reſpondent point a, or cut it in the reſpondent points a, b, accordingly as AB touches or cuts the circle (9. & 1. Cor. 3.). In the firſt inſtance AS is equal to aF, in the ſecond the ſegments AS, SB, are reſpectively equal to the ſegments aF, Fb (6. Cor. 3.). Therefore the ſquare of AS is equal to the ſquare of aF, and the rectangle ASB to the rectangle aFb.

57, 58. Case 2. If AB be not parallel to the directrix, but with its focal reſpondent FP meet the directrix in P; then for the ſame reaſon as in the 1. Caſe, will FP either touch the circle in the reſpondent point a, or cut it in the reſpondent points a, b; and I ſay, that the ſquare of AS, or the rectangle ASB, ſhall be to the ſquare of aF, or the rectangle aFb, as the ſquare of SP is to the ſquare of aP, or the rectangle aPb.

Join Sa, FA, and Sb, FB, which are parallel (19. Def.). AS is to aF as SP is to aP, therefore when AB touches the ſection,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section, the square of AS is to the square of aF as the square of SP is to the square of aP. But when AB cuts the section, because AS is to aF as SP is to aP, and BS is to bF as SP is to bP, therefore by the composition of these ratios, the rectangle ASB is to the rectangle aFb as the square of SP is to the rectangle aPb. FIG.

PROP. 22.

If two parallel right lines both touch, or if one touch and the other cut the same section in two points, or if both cut the same section or the opposite hyperbolas in two points, and any two points, one in each, be assumed, round which the generating circles are described; then,

In the First instance, the squares of the segments of the touching lines, intercepted between the points assumed and the points of contact;

In the Second, the square of the segment of the touching line and the rectangle under the segments of the cutting line, intercepted between the points assumed and the section;

In the Third, the rectangles under the segments of the cutting lines, intercepted between the points assumed and the section,

Shall be to each other as the rectangles under the segments of any two right lines drawn from the focus to cut the circles, viz. intercepted between the focus and each circle; Or, what is the same thing, as

FIG. as the squares of the right lines drawn from the focus to touch each circle.

59, 60. Let the parallels EG, AB, either both touch the same section
61, 62. or the opposite hyperbolas in E, A, or the one EG touch the
63, 64. section in E, the other AB cut the same section in A, B; or both of them cut the same or opposite sections, viz. in E, G, and A, B; and round any two points in each parallel, viz. O in EG, S in AB, let the generating circles be described, and FA, FE, touching the circles in A, E, or FAB, FGE, cutting them in A, B, and G, E. Then in the

1. CASE. When both the parallels touch the section or sections.

59, 60. I say, that the square of OE is to the square of SA as the square
59. of FE is to the square of FA.

If EG, AB, be each parallel to the directrix, their focal respondents FE, FA, will also be parallel to the directrix, will touch the circles in the respondent points E, A (9.), and the square of OE be equal to the square of FE, and the square of SA to the square of FA (21.). Therefore because equal magnitudes are proportional magnitudes, the square of OE is to the square of SA as the square of FE is to the square of FA.

60. If the parallels EG, AB, meet the directrix in R, P, the focal respondents FR, FP, will, as before, touch the circles in the respondent points E, A; and the square of OE will be to the square of FE as the square of OR is to the square of RE, and the square of SA be to the square of FA, as the square of SP is to the square of PA (21.). But the square of OR is to the square of RE as the square of SP is to the square of PA (2. COR. LEM. 14.). Therefore ex æquo, and by alternation, the square of OE is to the square of SA as the square of FE is to the square of FA.

61, 62. 2. CASE. If one of the parallels AB cut the section in A, B,
63. while EG touches it in E; I say, that the square of OE is to the rectangle ASB as the square of FE is to the rectangle AFB.

The

The ſame things remaining, if EG, AB, be parallel to the directrix, the ſquare of OE is equal to the ſquare of Fe, and the rectangle ASB to the rectangle aFb (21.). Therefore the ſquare of OE is to the rectangle ASB as the ſquare of Fe is to the rectangle aFb. FIG. 61.

If EG, AB, meet the directrix in R, P, their focal reſpondents FR, FP, will, the one touch its circle in the reſpondent point e, the other cut its circle in the reſpondent points a, b. But the ſquare of OE is to the ſquare of Fe as the ſquare of OR is to the ſquare of Re; alſo the rectangle ASB is to the rectangle aFb as the ſquare of SP is to the rectangle aPb (21.), and the ſquare of OR is to the ſquare of Re as the ſquare of SP is to the rectangle aPb (2. COR. LEM. 14.); therefore ex æquo and by alternation, the ſquare of OE is to the rectangle ASB as the ſquare of Fe is to the rectangle aFb. 62, 63.

3. CASE. If both the parallels EG, AB cut the ſection or ſections in E, G, and A, B; I ſay, that the rectangle EOG is to the rectangle ASB as the rectangle eFg is to the rectangle aFb. 64.

If EG, AB, be parallel to the directrix, for the ſame reaſon as in CASE 2., the rectangles EOG, ASB are reſpectively equal to the rectangles eFg, aFb; therefore, the rectangle EOG is to the rectangle ASB as the rectangle eFg is to the rectangle aFb.

If the parallels meet the directrix in R, P. Then it has been ſhewn in CASE 2. that the rectangle ASB is to the rectangle aFb as the ſquare of SP is to the rectangle aPb, and for the ſame reaſon the rectangle EOG muſt be to the rectangle eFg as the ſquare of OR is to the rectangle eRg; while the ſquare of OR is to the rectangle eRg as the ſquare of SP is to the rectangle of aPb (2. COR. LEM. 14.); therefore ex æquo and by alternation, the rectangle EOG is to the rectangle ASB as the rectangle eFg is to the rectangle aFb.

COR. 1. If two parallels, meeting each an aſſymptote of the hyperbola, meet the ſame or oppoſite ſections in two points, the

FIG. rectangles under the ſegments intercepted between the ſection or ſections and the aſſymptotes are equal between themſelves.

81. Let the parallels AB, EG, meeting the ſection or ſections in A, B, and E, G, meet each an aſſymptote of the hyperbola in S, O. I ſay, that the rectangles ASB, EOG, are equal between themſelves.—The ſame things remaining as in the PROP., the generating circles deſcribed round S, O, paſs through M or m, the concourſe of the aſſymptotes with the directrix (8. COR. DEF.), and FM, Fm, are perpendicular to the aſſymptotes (19. COR. DEF.), are equal between themſelves (19. COR. DEF.), and being perpendicular to SM, OM or Cm, they touch the circles in M or m. But the rectangles ASB, EOG, by this PROP., are as the ſquares of theſe equal tangents FM or Fm, therefore the rectangle ASB is equal to the rectangle EOG.

COR. 2. If two parallels, the one touching, the other cutting either hyperbola, meet either of the aſſymptotes, the ſquare of the touching line and the rectangle under the ſegments of the cutting line, intercepted between the ſection and the aſſymptote, ſhall be equal between themſelves.

This is demonſtrated in the ſame terms as the preceding corollary.

COR. 3. Hence if a right line meet an hyperbola or the oppoſite hyperbolas in two points, and meet each aſſymptote, the rectangle under its ſegments between the ſection and one aſſymptote is equal to the rectangle under the ſegments between the ſection and the other aſſymptote.

COR. 4. Hence if a right line meet an hyperbola or the oppoſite hyperbolas in two points, the rectangle under its ſegments between the ſection or ſections and either aſſymptote is equal to the ſquare of the ſemidiameter parallel to the right line.

This when the right line meets the oppoſite hyperbolas, and the ſemidiameter is tranſverſe, is no more than a caſe of the 1. COR.; and when the right line cuts the ſame ſection, it is a caſe of the 2. COR., for the ſegment of the touching line parallel to the cutting line, intercepted between the point of contact and an aſſymptote is equal

equal to the semidiameter secunda, parallel to the same. VIDE NOTE to 15. DEF. FIG.

COR. 5. If two parallel right lines do either both cut, or the one touch and the other cut a parabola, or an hyperbola or the opposite hyperbolas, or each touch an opposite hyperbola, and in the case of the parabola meet a diameter of the section, in the case of the hyperbola meet a parallel to either assymptote of the section; the rectangles under the segments of the cutting lines, or the square of the touching line and the rectangle under the segments of the cutting line, or the squares of the touching lines, intercepted between the sections and the diameter or parallel, shall be to each other, as the abscisses of the diameter or of the parallel to an assymptote, intercepted between each section and the two parallel right lines.

All other things remaining the same as in the PROP., let HD a diameter of a parabola, or a parallel to an assymptote of an hyperbola, meet the parallels EG, AB in O, S, and each section in D. I say, that the rectangle EOG or the square of OE is to the rectangle ASB or the square of SA as the absciss OD is to the absciss SD. 82, 83, 84.

Let HD meet the directrix in H. The generating circles described round O, S, pass through H (6. & 8. COR. DEF.), therefore HF, being joined, will meet each circle again. Let it meet them in the points D, d, which will in each circle be respondent to D (3.), and therefore FD, OD, Sd, being joined, will be parallel to each other. But by this PROP. the rectangle EOG or the square of EO will be to the rectangle ASB or the square of SA (as the rectangle HFD is to the rectangle HFd, viz. as FD is to Fd (1. e. 6.), viz. on account of the parallels,) as OD is to SD.

COR. 6. Hence therefore if in the case of the parabola the two parallels be ordinately applied to the diameter which they meet, the squares of the semi-ordinates shall be to each other as the abscisses.

FIG. For the rectangles under the ſegments of the parallels are then the ſame with the ſquares of the ſemi-ordinates, becauſe the parallels are biſected by the diameter to which they are ordinately applied (15.).

PROP. 23.

If two right lines meeting each other, do both touch, or the one touch and the other cut, or both cut the ſame or oppoſite ſections, the ſquares of the touching lines, or the ſquare of the touching line and the rectangle under the ſegments of the cutting line, or the rectangles under the ſegments of the two cutting lines, intercepted between their concourſe and the ſection or ſections, are as the focal lines parallel to them.

65, 66, 67. Let the right lines AB, DE, meeting each other in S, either both touch in the points A, D; or the one touch in A, the other cut in the points D, E; or both cut, the one in A, B, the other in D, E, the ſame ſection or the oppoſite hyperbolas, and through the focus F be drawn GFH parallel to AB, meeting the ſection in G, H, and IFL parallel to DE, meeting the ſection in I, L; I ſay, that the ſquare of AS is to the ſquare of DS, or the ſquare of AS is to the rectangle DSE, or the rectangle ASB is to the rectangle DSE, as GH is to IL.

Round S deſcribe the generating circle, and from F draw a right line cutting this circle in Q, q; alſo biſect GH in O, IL in P. Becauſe GH, IL, are drawn through the focus, and are biſected in O, P, the generating circles deſcribed round O, P, would paſs through G, H, and I, L (2. Cor. Def.), and becauſe GH is alſo

parallel

parallel to AB, therefore the square of AS or the rectangle ASB is to the rectangle QFq as the rectangle GOH or the square of GO is to the rectangle GFH (21.). But the rectangle IFL is to the rectangle GFH (as IL is to GH (COR. 20.), viz.) as IP is to GO; and IP is to GO as the rectangle under IP, GO, is to the square of GO (1. e. 6.), therefore the rectangle under IP, GO, is to the square of GO as the rectangle IFL is to the rectangle GFH. But it has been shewn, that the square of AS or the rectangle ASB is to the rectangle QFq as the square of GO is to the rectangle GFH, therefore ex æquo and by alternation, the square of AS or the rectangle ASB is to the rectangle QFq as the rectangle under IP, GO is to the rectangle IFL. Again, the square of DS or the rectangle DSE is to the rectangle QFq as the rectangle IPL or the square of IP is to the rectangle IFL (21.); wherefore, ex æquo, the square of AS or the rectangle ASB is to the square of DS or the rectangle DSE (as the rectangle under IP, GO is to the square of IP viz. as GO is to IP (1. e. 6.), viz. by doubling the consequents) as GH is to IL.

COR. 1. If a right line touching a conic section meet two parallels, of which, both also touch, or while one touches, the other cuts, or both cut the same section or the opposite hyperbolas, the squares of the segments of the touching parallels, or the square of the segment of the touching parallel and the rectangle under the segments of the parallel which cuts, or the rectangles under the segments of the parallels which cut the section, viz. in each case, the segments intercepted between the section and the touching line falling upon the parallels, shall be as the squares of the segments of the touching line, intercepted between its point of contact and the parallels.

Let the right line GH touching a conic section in G meet in the points S, R, the parallel right lines AB, DE, either both touching in A, D; or the one AB touching in A, the other cutting in D, E; or both cutting, the one in A, B, the other in D, E, the same section or opposite hyperbolas; I say, that the squares

of

FIG. 71, 72, 73.

68, 69,

FIG. of AS, DR, or the square of AS and the rectangle DRE; or the rectangles ASB, DRE, are as the squares of GS, GR.

Through the focus F draw AB parallel to AB or DE, and GH parallel to GH, meeting the section or sections in A, B, and G, H. The square of AS is to the square of GS, or the rectangle ASB is to the square of GS (as AB is to GH, viz.) as the square of DR is to the square of GR, or as the rectangle DRE is to the square of GR. Therefore by alternation, the corollary is inferred.

71, 72, 73. Cor. 2. If a right line cutting a conic section or the opposite hyperbolas meet two parallels, of which both touch; or one touches, and the other cuts; or both cut the same section or sections; the squares of the segments of the touching parallels, or the square of the segment of the touching parallel and the rectangle under the segments of the parallel which cuts, or the rectangles under the segments of the cutting parallels, shall be in the same proportion as the rectangles under the segments of the cutting line; understanding the segments to be as in the preceding.

This is inferred in the same manner as the preceding Cor.

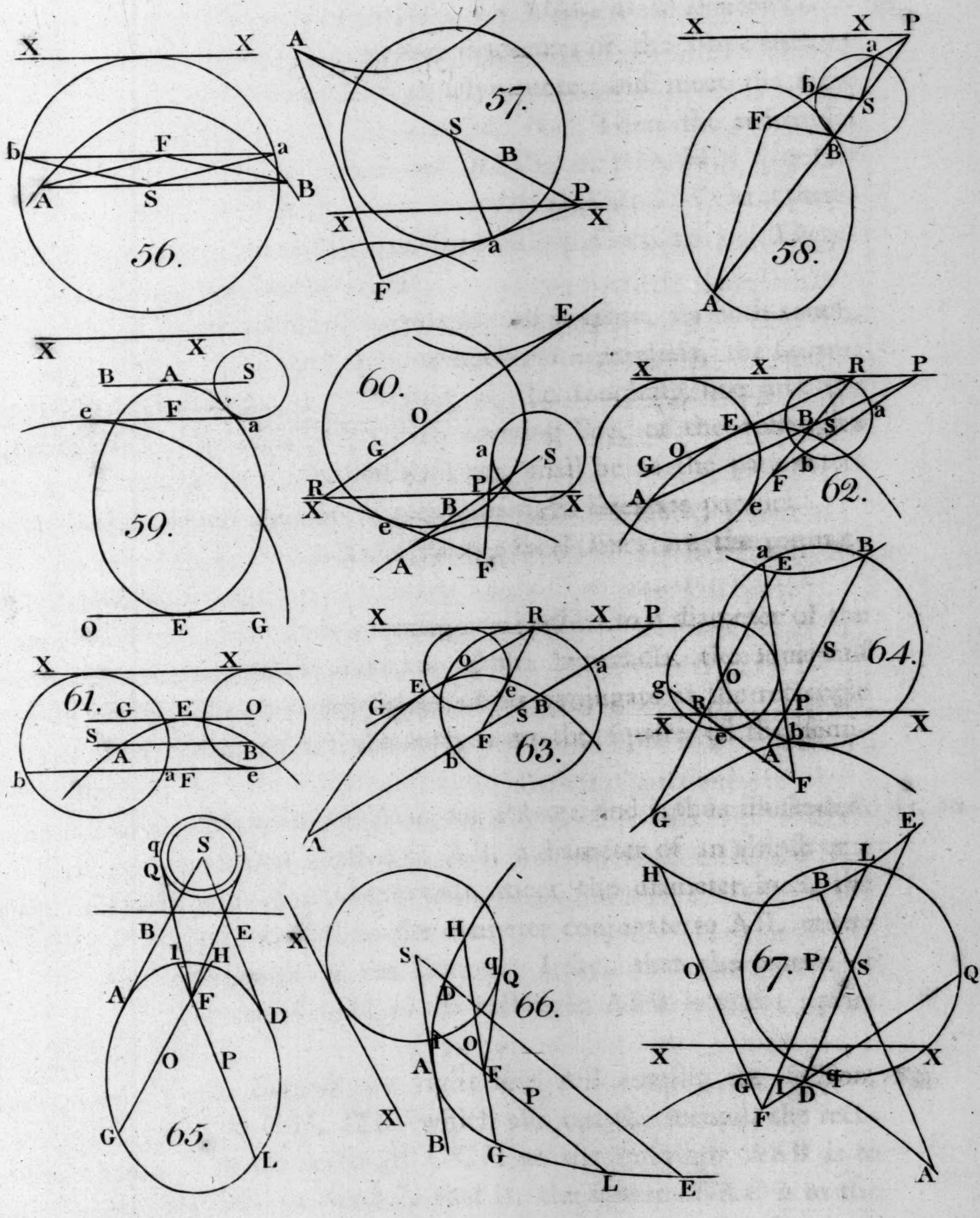
74, 75. Cor. 3. If two right lines meeting each other, do both touch, or the one touch and the other cut, or both cut the same section or opposite sections, and two other right lines parallel to the former pair fall upon the section or sections in like manner; the squares of the first pair if they both touch, or the square of the one which touches and the rectangle under the segment of the one which cuts, or the rectangles under the segments of each if both cut, shall be in the same proportion as the like squares or rectangles of the other pair.

For the magnitudes of the one pair and the magnitudes of the other pair are proportional to the same focal lines.

Cor. 4. The focal lines or focal ordinates parallel to two diameters of the ellipse or hyperbola, are in the duplicate proportion of the semidiameters, or of the diameters.

This in the case of the ellipse is the Prop. itself in other words; for the diameters mutually bisecting each other in the centre, the rectangle under the segments of each diameter becomes the square

of


56.
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of the ſemidiameter, and therefore the focal parallels are as the FIG.
ſquares of the ſemidiameters, or of the diameters.

But in the caſe of the hyperbola, the COR. is thus illuſtrated. 86.
Let ED, MN, parallel to any two diameters of the hpperbola be drawn through any point O in an aſſymptote, and meet the ſame or oppoſite ſections in D, E, and M, N. Then the rectangles EOD, MON are as the focal lines parallel to ED, MN (by this PROP.), while the rectangles are reſpectively equal to the ſquares of the ſemidiameters parallel to ED, MN (4. COR. 22.). Therefore the COR. immediately follows.

COR. 5. If two right lines meeting each other, do both touch, or one touch and the other cut, or both cut a parabola, the ſquares of the touching lines, or the ſquare of the touching line and the rectangle under the ſegments of the cutting line, or the rectangles under the ſegments of the cutting lines, ſhall be as the parameters of the diameters to whoſe ordinates the right lines are parallel.

This is alſo the PROP. itſelf, for the focal lines are the parameters (24. DEF.).

COR. 6. If a right line be ordinately applied to a diameter of the ellipſe, or to a tranſverſe diameter of the hyperbola, the ſquare of the diameter ſhall be to the ſquare of its conjugate as the rectangle under the abſciſſes of the diameter is to the ſquare of the ſemi-ordinate.

This immediately follows from the PROP., and is thus illuſtrated. 85, 86.
Let DE ordinately applied to AB, a diameter of an ellipſe or a tranſverſe diameter of an hyperbola, meet the diameter in S, the ſection in D, E, and GH be the diameter conjugate to AB, meeting it in C the centre of the ſection. I ſay, that the ſquare of AB is to the ſquare of GH as the rectangle ASB is to the ſquare of SD or SE.

In the ellipſe, becauſe the right line AB cutting the ſection, 85.
meets the parallels GH, DE, which alſo cut the ſection, the rectangle ACB is to the rectangle GCH as the rectangle ASB is to the rectangle DSE (2. COR.), that is, the ſquare of AC is to the ſquare of GC as the rectangle ASB is to the ſquare of SD.

In

FIG. In the hyperbola, draw AI touching the section in the vertex A,
86. and meeting either assymptote in I, also draw KIL parallel to AB meeting the opposite sections in K, L (7.), and through the focus draw the focal lines AFB, DFE, parallel to AB, DE. The rectangle KIL is equal to the square of AC (4. COR. 22.), and AI is equal to GC or HC (NOTE. 15. DEF.). But by this PROP., the rectangle KIL is to the square of AI, viz. the square of AC is to the square of GC, as AB is to DE; also, the rectangle ASB is to the rectangle DSE or the square of DS in the same ratio of AB to DE. Therefore, ex æquo, the square of AC is to the square of GC as the rectangle ASB is to the square of DS. And in whatever proportion AC, GC, are, in the same proportion are their doubles AB, GH.

COR. 7. Hence it follows, that if an ordinate to a diameter secunda of an hyperbola be drawn, the square of the diameter is to the square of the transverse diameter conjugate thereto, as the sum of the squares of the semidiameter secunda and of the segment intercepted between the centre and ordinate is to the square of the semi-ordinate.

86. If DP ordinately applied to the diameter secunda GH meet it in P; I say, that the square of CG shall be to the square of AC as the squares of CG, CP together are to the square of DP.

The same things remaining as in the last, the square of CG is to the square of AC as the square of SD or CP is to the rectangle ASB, therefore, componendo, the square of CG is to the square of AC as the squares of CG, CP together are to the square of CS or DP.

PROP. 24.

If two parallel right lines be inscribed in a conic section, and either cut or touch the section, the point of

of contact being confidered as a double point, and anfwering to the two points, in which a right line cutting the fection meets it; and if from two of their terms, one in each, be inflected two right lines to a point in the fection, meeting each the alternate parallel; the rectangle under the fegments of the parallels intercepted between the inflected lines and the two remaining terms of the parallels fhall be to the fquare of the right line joining either of thefe terms and the remaining term in the other parallel from which the inflected right line is drawn, as the focal line parallel to the infcribed parallels is to the focal line parallel to the joining right line. FIG.

CASE 1. When neither of the inflected right lines touch the fection.

Let AB, CD, parallel to each other, be infcribed in a conic fection, and either cut or touch the fection; from two alternate terms A, C, inflect AE, CE, to a point E in the fection, meeting CD in F, AB in G, and join AD, BC. I fay, that the rectangle under DF, BG, is to the fquare of AD or BC as the focal line parallel to AB is to the focal line parallel to AD or BC. 90, 91, 93.

Draw EK parallel to AB, CD, meeting the fection again in K, and AD, BC in I, H. Becaufe of the parallels, BG is to EH (as BC is to CH, viz.) as AD is to DI; and by alternation, BG is to AD as EH is to DI. But alfo, DF is to AD as EI is to AI. Therefore, compounding thefe two ratios, the rectangle under DF, BG, is to the fquare of AD as the rectangle HEI is to the rectangle AID. But HE is equal to IK (9. COR. 15.), therefore the rectangle HEI, is equal to the rectangle EIK, and the rectangle under DF, BG, is to the fquare of AD (as the

FIG. rectangle EIK is to the rectangle AID, viz.) as the focal line parallel to EI or AB is to the focal line parallel to AD (23.).

91, 93. The proposition is the same, and the demonstration is conducted in the same words, if AB touch the section, or CD also, the points of contact being considered as double points, and represented in the Fig. by the double letters A, B, and C, D.

CASE 2. When either of the inflected right lines touches the section.

92. If AE touch the section, in which case the point E falls in A, CE becomes the same with CA, the points G, I, also fall into A, and the points H, K, into B; yet I say, that the rectangle under DF and BG or AB is to the square of AD or BC as the focal line parallel to AB is to the focal line parallel to AD or BC.

Draw DM parallel to AE or AF, meeting the section again in M, and AB in V, also join MB meeting CD in N. The square of AF is to the rectangle DVM as the rectangle DFC is to the rectangle AVB (3. COR. 23). But because of the parallels, AF is equal to DV, and DF to AV, therefore (1. e. 6.), AF is to VM as FC is to VB, and because the angle AFC is equal to the angle MVB, the bases AC, MB will be parallel (6. e. 6. & 27. e. 1.). Consequently ABNC is a parallelogram, as is also AFDV, and AB being equal to CN, DF to AV, the rectangle under AB, DF will be equal to the rectangle under CN, AV. But because from the terms B, D, of the inscribed parallels AB, DC, are inflected to M the right lines BM, DM, and meet the parallels in N, V; therefore, by the preceding Case, the rectangle under CN, AV, viz. under AB, DF, will be to the square of AD or BC as the focal line parallel to AB is to the focal line parallel to AD or BC.

The demonstration of this Case would be the same, if CD had touched the section.

92. COR. 1. If two parallels be inscribed in a conic section, and a right line touching the section at either term of one of the parallels

meet the other; the rectangle under the parallel whose term is the point of contact and either segment of the other, intercepted between one of its concourses with the section and the tangent, shall be to the square of the right line joining that concourse and the point of contact, or to the square of the right line joining the other two terms of the parallels, as the focal line parallel to the inscribed parallels is to the focal line parallel to the right line joined. FIG.

This is merely the second case of the proposition.

COR. 2. If two parallels be inscribed in a conic section, and from two of their terms, one in each, be inflected two right lines to a point in the section, and in like manner from the two remaining terms to any other point in the section, the rectangles under the segments of the parallels, intercepted between each pair of the inflected lines and that term in each parallel from which the respective pair is not inflected, shall be equal between themselves.

The same things remaining as in the Prop., from the two remaining terms B, D, draw BL, DL, to any point L in the section, meeting CD, AB, in F, G; then by the Prop., the rectangle under CF, AG, has to the square of AD the same proportion which the rectangle under DF, BG, has to the square of AD, and therefore the rectangle under DF, BG, is equal to the rectangle under CF, AG. 90, 91. 92.

COR. 3. If two parallel right lines be inscribed in a conic section, and the right lines joining their terms be drawn, the squares of these right lines so drawn shall be as the focal lines parallel to them.

The same things remaining, the rectangle under BG, DF is to the square of AD as the focal line parallel to AB is to the focal line parallel to AD; and the same rectangle under BG, DF is to the square of BC as the focal line parallel to AB is to the focal line parallel to BC. Therefore, ex æquo, the square of AD is to the square of BC as the focal line parallel to AD is to the focal line parallel to BC. 90, 91. 92.

COR. 4. If a right line touch a conic section, and parallel to it be inscribed in the section a right line cutting it, the squares of the

 right

FIG. right lines joining the point of contact and the terms of the parallel cutting the section, shall be as the focal lines parallel to them.

This is inferred in the same manner as the last.

Cor. 5. If two right lines touching a conic section meet each other, and from each point of contact a right line be drawn parallel to the alternate tangent, meeting again the section, the rectangles under each parallel and the segment of the tangent to which it is parallel, intercepted between the contact and the concourse of the tangents, shall be to each other as the focal lines parallel to the tangents.

95. Let AF, CF, touching a conic section in A, C, meet each other in F, and AB, CL, parallel to CF, AF, meet the section again in B, L; I say, that the rectangles under AB, CF, and CL, AF, are as the focal lines parallel to CF, AF, or to AB, CL.

Join AC. By Case 2. of this Prop. the rectangle under AB, CF is to the square of AC as the focal line parallel to AB is to the focal line parallel to AC; and the rectangle under CL, AF, is to the square of AC as the focal line parallel to CL is to the focal line parallel to AC. Therefore, ex æquo, the rectangle under AB, CF is to the rectangle under CL, AF as the focal line parallel to AB is to the focal line parallel to CL.

Cor. 6. The same things remaining as in the last Corollary, the right lines drawn parallel to the tangents are in the same proportion as the tangents to which they are parallel.

95. Because AF, CF, touching the section, meet each other in F, the square of CF is to the square of AF as the focal line parallel to CF or AB is to the focal line parallel to AF or CL (23.). Therefore, ex æquo, the rectangle under AB, CF is to the rectangle under CL, AF as the square of CF is to the square of AF, and consequently AB is to CL as CF is to AF (1. e. 6.).

Cor. 7. If from the vertices of a diameter of an ellipse, or a transverse diameter of an hyperbola, two right lines be inflected to any point in the section, and each meet the tangent to the section at the alternate vertex, the rectangle under the segments of the tangents,

tangents, intercepted between the vertices and the inflected lines, FIG.
ſhall be to the ſquare of the diameter as the focal line parallel to the tangents is to the focal line parallel to the diameter.

This is merely that caſe of the Prop., when AB, CD, both 93.
touch the ſection, in which caſe AD or BC becomes the diameter (12.).

Cor. 8. If a right line, touching an ellipſe or hyperbola, meet two conjugate diameters, the rectangle under the ſegments of the touching line, intercepted between the contact and the diameters, is equal to the ſquare of the ſemidiameter parallel to the touching line.

Let a right line ST touching an ellipſe or hyperbola in C, meet 94.
two conjugate diameters OT, OS in T, S, and OZ be the ſemidiameter parallel to ST; I ſay, that the rectangle SCT is equal to the ſquare of OZ.

Draw the diameter COA, and CE ordinately applied to OT, viz. parallel to OS, meeting OT in V, and the ſection in E; join AE, meeting ST in F, and draw AG touching the ſection in A, or parallel to ST, and meeting OS in R, CE in G. In the triangle ACE, becauſe AC is biſected in O, and CE in V (15.), AE is parallel to OV. Alſo in the triangles ACG, ACF, becauſe AC is biſected in O, AG, CF, will be biſected in R, T, and AG is double to GR, that is, to SC, and CF is double to CT. But the rectangle under AG, CF, is to the ſquare of AC (as the focal line parallel to AG or CF is to the focal line parallel to AC (7. Cor.), viz.) as the ſquare of OZ is to the ſquare of AO (4. Cor. 23.). Wherefore the rectangle under the halves being in the ſame proportion to the ſquare of the half, the rectangle SCT will be to the ſquare of AO as the ſquare of OZ is to the ſquare of AO, viz. the rectangle SCT will be equal to the ſquare of OZ.

Cor. 9. If a right line touching an ellipſe or hyperbola meet a diameter, and from the point of contact be drawn an ordinate to the diameter, the rectangle under the ſegments of the diameter, intercepted between the centre and tangent, the centre and ordinate, ſhall be equal to the ſquare of the ſemidiameter.

Let

FIG. Let a right line DE touching an ellipse or hyperbola in D, meet
96, 97. any diameter ACB in E, and from D be drawn DF ordinate to AC, meeting it in F, then C being the centre of the section; I say, that the rectangle ECF is equal to the square of AC.

Let ED meet ab, the diameter conjugate to AB, in e, and CG be semidiameter parallel to ED. Then the rectangle AEB is to the square of ED (as the focal ordinate parallel to AB is to the focal ordinate parallel to ED (23.), viz.) as the square of AC is to the square of CG (4. Cor. 23). But the rectangle $\text{D}e$ is equal to the square of CG (8. Cor.), therefore and by alternation, the rectangle AEB is to the square of AC (as the square of ED is to the rectangle $\text{ED}e$, viz.) as ED is to $\text{D}e$. Wherefore, componendo, or dividendo, the square of EC is to the square of AC (as $\text{E}e$ is to $\text{D}e$, viz.) as EC is to CF, AC is therefore a mean proportional between EC, CF, or the rectangle ECF is equal to the square of AC.

Cor. 10. If a right line touch an ellipse or hyperbola, and from the point of contact a perpendicular be drawn, meeting either axis of the section, also from the centre of the section a perpendicular be drawn to the touching line; the semi-conjugate axis shall be a mean proportional between the perpendiculars.

97. If the conjugate diameters in the last Corollary be the conjugate axes, then every thing remaining the same, viz. $a\text{C}b$ being the conjugate axis to ACB, if DP perpendicular to ED meet either axis AB in P, and CI be drawn perpendicular to ED; I say, that $a\text{C}$ shall be a mean proportional between PD, CI.—Draw $\text{D}f$ ordinately applied to the axis ab.

Because of the parallels, the angle $e\text{CI}$ is equal to the angle PDF, and the angles at I, F, are right; therefore the triangles PDF, $e\text{CI}$, are equiangular, and PD is to DF or $\text{C}f$ as $e\text{C}$ is to CI. But by the last Corollary, $\text{C}f$ is to $a\text{C}$ as $a\text{C}$ is to $e\text{C}$; therefore ex æquo perturbatè, PD is to $a\text{C}$ as $a\text{C}$ is to CI.

If

If the perpendicular meet the axis AC in P, it will in like manner be shewn that PD is to AC as AC is to CI. FIG.

COR. II. If a right line touch an ellipse or hyperbola, and a perpendiculer thereto at the point of contact meet each of the axes, the segments of the perpendicular intercepted between the point of contact and each axis, shall be alternately in the duplicate ratio of the semi-axes or axes.

The same things remaining as in the last COR., the square of AC is equal to the rectangle under PD, CI, and the square of AC is equal to the rectangle under PD, CI; therefore the square of AC is to the square of AC (as the rectangle under PD, CI is to the rectangle under PD, CI, viz.) as PD is to PD (4. e. 1.).

PROP. 25.

If two right lines be inscribed in a conic section, and either cut or touch the section, the point of contact being considered as a double point, in like manner as in the last, and if from two of their terms, one in each, two right lines be inflected to a point in the section, and from the two remaining terms two other right lines be inflected to any other point in the section; the right line joining the intersections of the inflected right lines, viz. the intersection of one of each pair with the alternate one of the other pair, shall verge to the concourse of the inscribed right lines, viz. it shall pass through their concourse, if they do meet, and be parallel to them, if they be parallel between themselves.

Let

FIG. Let AB, CD, be inscribed in a conic section, and from the terms
98, 99. A, C, let AE, CE, be inflected to a point E in the section, also
100, 101. from the remaining terms B, D, be inflected BL, DL to any other
102, 103. point L in the section; and let AE, DL, meet in O, CE, BL,
104, 105. meet in P; I say, that OP, if joined, shall verge to the concourse
of AB, CD.

99, 100. Case 1. When AB, CD, are parallel to each other; I say,
101, 102. that OP shall also be parallel to AB or CD.

Describe any generating circle, whose centre is S, and therein inscribe ab, cd, the focal correspondents to AB, CD, also ae, dl, the focal correspondents to AE, DL, meeting each other in o, and ce, bl, the focal correspondents to CE, BL, meeting in p. The points o, p, will be correspondent to O, P (4.), and op, being joined, shall be the focal correspondent to
100. OP.—Then if AB, CD, be parallel also to the directrix, ab, cd, shall also be parallel to the directrix (18. Def.). Therefore because in the circle are inscribed the two parallels ab, cd, and from the terms a, c, are inflected ae, ce, to a point e in the circumference, and from the remaining terms, b, d, are inflected bl, dl, to another point l in the circumference, the right line op, which joins the concourse o of ae with dl, and the concourse p of ce with bl, shall be parallel to ab or cd (Lem. 12.), viz. be parallel to the directrix. Wherefore the conic line OP, to which op is the focal correspondent, shall also be parallel to the directrix, viz. be parallel to AB or CD.

99, 101. If AB, CD, meet the directrix, then being parallel to each other,
102. their focal correspondents ab, cd, will meet in the directrix (12. Cor. Def.). Let them meet in r, because ab, cd, are inscribed in a circle, and meet in r, and from two of their terms a, c, are inflected ae, ce, to a point e in the circumference, and from the remaining terms b, d, are inflected bl, dl, to another point l in the circumference, the right line op, which joins the concourse o of ae with dl, and p the concourse of ce with bl, shall pass

through

through R (LEM. 12.). Wherefore the three right lines AB, CD, OP, pertaining to the circle, having their common concourſe in the directrix, the right lines AB, CD, OP, which are their conic reſpondents, ſhall be parallel between themſelves (12. COR. DEF.). FIG.

CASE 2. When AB, CD, are not parallel, but meet in R; I ſay, that OP verges towards R, viz. that O, P, R, are in one ſtrait line. 98, 103. 104, 105.

Every thing being conſtructed the ſame as in the laſt, every thing will remain the ſame, except that R, the concourſe of AB, CD, is not in the directrix. For the ſame reaſon, as in the preceding caſe, the points O, P, R, are in one ſtrait line. Join OR, PR, and draw SQ, the conic correſpondent to thoſe reſpondents, of which OPR is the focal correſpondent. Then OR is the focal correſpondent to OR, and PR is the focal correſpondent to PR (2. COR. 4.); but OR, PR, are one and the ſame right line, therefore OR, PR, their conic reſpondents, are parallel to one and the ſame right line (18. DEF.), viz. to SQ, and conſequently being drawn through the ſame common point R, they are one and the ſame right line.

COR. If two right lines be inſcribed in a conic ſection, and the right lines joining their terms, be drawn, the two concourſes of theſe right lines, and the concourſes of the tangents to the ſection at the terms of each of the inſcribed lines, ſhall be in the ſame right line.

Let AE, DL, be inſcribed in a conic ſection, and AL, DE, meeting in P, and AD, EL, meeting in Q, be joined. Alſo, let AR, ER, touching the ſection in A, E, meet in R, and DO, LO, touching the ſection in D, L, meet in O; I ſay, that the four points P, Q, R, O, ſhall be in one ſtrait line. 106.

Becauſe the right lines AR, ER, touching the ſection, meet in R, and from the points of contact A, E, are inflected AD, ED, and alſo AL, EL, to the points D, L, in the ſection, the interſections Q, P, of AD, EL, and of ED, AL, will be in the ſame

FIG. right line with R. For the same reason, DO, LO, touching the section in D, L, and DA, LA, being inflected to the point A in the section, and DE, LE to the point E in the section, the three points Q, P, O, will be in one strait line. Therefore the four points Q, P, R, O, are in one and the same right line.

PROP. 26.

If a quadrilateral be inscribed in a conic section, or in the opposite hyperbolas, and from any point in the section or in either hyperbola be drawn two right lines parallel to two adjacent sides of the quadrilateral, the rectangles under the segments of the right lines between the point in the section and the opposite sides of the quadrilateral shall be proportional to the focal ordinates parallel to the two adjacent sides of the quadrilateral.

108. CASE I. When two sides of the quadrilateral are parallel.

109. Let ABCD be a quadrilateral inscribed in a conic section, or in the opposite hyperbolas, having the side AB parallel to CD; and parallel to two adjacent sides AB, BC of the quadrilateral, let EPF, EHL, be drawn from any point E in the section or sections, meeting each the opposite sides of the quadrilateral in P, F, and H, L; I say, that the rectangles FEP, HEL, are as the focal lines parallel to AB, BC.

The right line EPF either meets the section again in a point R, or touching the section, the points E, R, coincide; and in either case RF is equal to EP (9. COR. 15.), and the rectangle EFR is equal to the rectangle FEP. Again, because EB, EC, are parallelograms, BF is equal to EH, and FC to EL, and the rectangle

rectangle BFC is equal to the rectangle HEL. Wherefore the rectangle EFR is to the rectangle BFC as the rectangle FEP is to the rectangle HEL. But the rectangle EFR is to the rectangle BFC as the focal line parallel to ER or AB is to the focal line parallel to BC (23.); therefore ex æquo, the rectangle FEP is to the rectangle HEL as the focal line parallel to AB is to the focal line parallel to BC. FIG.

CASE 2. When two opposite sides of the quadrilateral are not parallel.

The same things remaining, except what is excepted; I say, that the same property obtains, viz. that the rectangles FEP, HEL, are as the focal lines parallel to AB, BC. 107.

Draw AK, DO, parallel to BC, the former meeting the section or an opposite hyperbola in K, and EF in I; the latter meeting AB in O; also join CK, meeting EH in G, and DO in Q. Because of the parallels, the triangles CGL, CQD, and AIP, DOA, are equiangular; wherefore GL is to DQ (as CL is to CD, viz.) as BH or EF is to BO, and, by alternation, GL is to EF as DQ is to BO. Again, AI or EH is to IP as DO is to AO. Therefore the ratio compounded of these two ratios will be, as the rectangle under EH, GL, is to the rectangle under EF, IP (so is the rectangle QDO to the rectangle AOB, viz.), so is the focal ordinate parallel to BC to the focal ordinate parallel to AB (1. CASE.). But for the same reason the rectangle HEG is to the rectangle FEI as the focal line parallel to BC is to the focal line parallel to AB. Therefore the whole or remaining rectangles HEL, FEP, will be in the same ratio of the focal lines parallel to BC, AB.

COR. 1. If in the 1. Case, when AB is parallel to CD, the right line drawn through E parallel to AB, touch the section in E, then EF becomes equal to EP (8. COR. 15.), and the square of EF or EP, being the same with the rectangle FEP, is to the 109.

 rectangle

FIG. rectangle HEL as the focal line parallel to AB is to the focal line parallel to BC.

Cor. 2. Hence the points A, B, C, D, remaining fixed, and the point E moving through the section or opposite sections, the ratio of the rectangles FEP, HEL, will be constantly the same.

107. Cor. 3. The three points A, B, C, and the point E, remaining fixed, and EF, EH being drawn parallel to AB, BC; if the point D move through the section, and AD, CD, being joined, meet EF, EH in P, L, the ratio of EP to EL shall be constantly the same.

For the ratio of the rectangles FEP, HEL, is constantly the same, viz. that of the focal lines parallel to AB, BC. But EF, HE, are constant, therefore the ratio of EP to EL is constantly the same.

This ratio is also given, if the point R, in which EF meets the section again, be given, or if EF touch the section in E, viz. if the points E, R, coincide; for the ratio is that of FR to CF.— By this Prop. the ratio of the rectangles FEP, HEL, is that of the focal lines parallel to AB, CD, and in the same ratio are the rectangles EFR, BFC (23.). Therefore the rectangle FEP is to the rectangle HEL as the rectangle EFR is to the rectangle BFC, and consequently, because HE is equal to BF, EP will be to EL as FR is to FC (1. e. 6.).

Cor. 4. If a quadrilateral be inscribed in a conic section, and from any point in the section be drawn four right lines to the four sides of the quadrilateral, the rectangle under any two of the right lines, which meet opposite sides, shall be to the rectangle under the other two in a given ratio.

110. Let the quadrilateral ABCD be inscribed in a conic section, or in the opposite hyperbolas, and from any point E in the section be drawn EH, EF, EL, EP, to meet the sides AB, BC, CD, DA; I say, that the rectangle FEP shall be to the rectangle HEL in a given ratio.

Draw

Draw EHL parallel to BC, meeting AB, cD, in H, L; and EFP parallel to AB, meeting BC, AD, in F, P. Then, because the sides of the quadrilateral and the right lines drawn from E, and consequently the right lines EHL, EFP, are given in position (31. DATA.), the triangles HEH, FEF, LEL, PEP, will be given in specie (28. & 43. DATA.). Wherefore the ratio of FE to EF, of EP to EP, of HE to EH, and of EL to EL, is given, and consequently the ratio compounded of these ratios is given, viz. of the rectangle FEP to the rectangle FEP, and of the rectangle HEL to the rectangle HEL. But by this proposition, the ratio of the rectangle FEP to the rectangle HEL is given; therefore the ratio of the rectangle FEP to the rectangle HEL is also given. FIG.

PROP. 27.

If from the focus of a conic section two right lines be drawn which meet the section towards the same parts, but on different sides of the axis, and are equal between themselves, they shall make equal angles with either axis. And more than two equal right lines cannot be drawn from the focus to meet the section towards the same parts of the focus.

Conversely. If from the focus two right lines be drawn which make equal angles with the axis, and one of them meet the section, the other shall also meet the section towards the same parts, and they shall be equal between themselves.

Let two right lines, FG, FH, drawn from the focus F of a conic section, and meeting the same section towards the same parts, 130, 131. 132.

be

FIG. be equal between themselves; I say, that FG, FH, shall make equal angles with either axis.

Let FA be the focal axis, and DCE the conjugate axis of the ellipse or hyperbola. Join GH, and draw GI, HT, perpendicular to the directrix. Then FG is to GI as FH is to HT (1. Cor. 1.); wherefore because FG is equal to FH, GI shall be equal to HT, and consequently GH is parallel to the directrix, and perpendicular to the axis FA. Because therefore FA is perpendicular to the base GH of the isosceles triangle GFH, it will bisect the angle GFH (Cor. 12. e. 1.), viz. FG, FH, make equal angles with FA. And in the ellipse and hyperbola, because GH, DE, are each perpendicular to FA, they are parallel between themselves, and GFH being an isosceles triangle, FG, FH, make equal angles with GH (5. e. 1.), viz. with DE (29. e. 1.).

I say, also, that no other right line equal to FG or FH can be drawn from F to meet the same section.—Let FV be any other right line drawn from F to meet the section in V, and from V draw VU perpendicular to the directrix. The point V is not in the right line GH, because a right line cannot meet a conic section in more than two points (5. Cor. 3.); wherefore VU is not equal to HT. But FV is to VU as FH is to HT (1. Cor. 1.), therefore FV is not equal to FH.

Conversely, if FG, FH, do make equal angles with FA, and one of them FG meet the section in G, the other shall also meet the section towards the same parts, and be equal to FG.—Draw GH parallel to the ordinates applied to FA, meeting AF in Q, and FH in H. Because the angles GQF, HQF, are right, the angles GFQ, HFQ equal between themselves, and the side FQ is common, the side GQ shall be equal to QH, and FG be equal to FH (26. e. 1.). Wherefore GH being bisected in Q, and the point G being in the section, the point H shall be in the section also (8. Cor. 15.). Wherefore FH does meet the section towards the same parts with the point G, and is equal to FG.

Cor. 1. If in a conic section a focal ordinate be drawn, another and only one other may be drawn, which is equal to it, and this other

other will make equal angles with either axis.—And conversely, two focal ordinates which do make equal angles with either axis, shall be equal between themselves. FIG.

Let GO passing through the focus F meet a conic section or the opposite hyperbolas in G, O; I say that another and only one other right line may be drawn through the focus, so as the part intercepted by the section or sections shall be equal to GO, and that this other shall make with either axis the same angles, which GO makes.

Draw FH making towards the opposite parts of the axis the angle AFH equal to the angle AFG. Because FG meets the section in G, FH will also meet in a point H, towards the same parts of the focus with G, and FH will be equal to FG. For the same reason will FH meet the section again or the opposite section in a point P, towards the same parts with O, and FP be equal to FO. Therefore the whole or remainder HP will be equal to the whole or remainder GO; and HP, GO make equal angles with FA. Moreover no other than HP can be drawn equal to GO, because no other than FH can be drawn equal to FG, and FP equal to FO.

Conversely. If the focal ordinates GO, HP, make equal angles with the axis, they shall be equal between themselves. For because FG, FH, make equal angles with FA, they are equal to each other, and for the same reason, are FO, FP, equal to each other; therefore the whole or remainder GO is equal to the whole or remainder HP.

COR. 2. In an ellipse and hyperbola, one diameter, and only one, may be drawn, which is equal to a given one, and these two equal diameters make equal angles with the axis.—And conversely, if two diameters of the ellipse and hyperbola do make equal angles with the axes, they shall be equal between themselves.

Let KL be a diameter of an ellipse or hyperbola, and C be the centre, F the focus, AB the focal axis. Towards different parts of the axis draw the diameter MN, making the angle NCA equal to the angle LCA. I say, that KL shall be equal to MN. 132.

Through

FIG. Through the focus F draw HP, GO parallel to KL, MN, meeting the section in H, P, and G, O. Because KL, MN, make equal angles with FA; HP, GO, which are parallel to them, will also make equal angles with FA, and consequently be equal between themselves (1. COR.). But HP, GO, are in the duplicate ratio of the diameters KL, MN (4. COR. 23.), therefore KL is also equal to MN. And no other diameter can be drawn equal to KL, because only two equal ordinates can be drawn through the focus.

COR. 3. If two right lines touching a conic section or the opposite hyperbolas, meet each other, and make equal angles with an axis, they shall be equal between themselves, the right lines joining their points of contact shall be ordinately applied to an axis, and the concourse of the tangents shall be in the axis itself.

And conversely, if two right lines touching a conic section meet each other, and the right line joining the points of contact be ordinately applied to an axis, the two right lines shall make equal angles with an axis, and their concourse shall be in the axis to which the ordinate is applied.

155, 159, 160. Let AE, CE, touching the same conic section, or opposite hyperbolas in A, C, meet each other in E, and make equal angles with either axis; I say, that AE is equal to CE, the right line AC joining the points of contact shall be ordinately applied to an axis, and the concourse E shall be in that axis.

Because AE, CE, make equal angles with the axes, the focal ordinates parallel to them will also make equal angles with the axes, and therefore be equal between themselves. But the focal ordinates are in the duplicate proportion of AE, CE (23.), therefore AE, CE, are also equal between themselves. Wherefore ACE is an isosceles triangle, and EI perpendicular to AC will bisect AC in I, and consequently will be the diameter to which AC is ordinately applied (1. COR. 18.). But because AC is ordinately applied at right angles, EI will be an axis of the section.

And conversely, if AE, CE, touching a conic section meet in E, and AC joining the points of contact be ordinately applied to an an axis;

axis; I ſay, that AE, CE do make equal angles with an axis, and that the concourſe E is in the axis to which AC is ordinately applied. FIG.

Draw the diameter of the ſection which paſſes through E. AC will be ordinately applied to EI (18.), and it is alſo ordinately applied to an axis; therefore EI is an axis of the ſection, and conſequently perpendicular to AC, and biſects it in I. Wherefore the angles AEI, CEI, and the right lines AE, EC, are equal between themſelves (4. e. 1.).

COR. 4. If two right lines, the one touching, the other cutting a conic ſection, or the oppoſite hyperbolas, meet each other, and the right lines make equal angles with the axes; the ſquare of the touching line ſhall be equal to the rectangle under the ſegments of the cutting line, intercepted between the ſection or ſections and the concourſe.

And converſely, if the ſquare of the one be equal to the rectangle under the ſegments of the other, the right lines ſhall make equal angles with the axes.

COR. 5. If two right lines, each cutting the ſection or oppoſite hyperbolas, meet each other, and make equal angles with an axis, the rectangles under the ſegments of each intercepted between the ſection or ſections and the concourſe, ſhall be equal to each other.

And converſely, if the rectangles be equal, the right lines ſhall make equal angles with the axes.

Theſe two corollaries are directly inferred from the PROP., as in the 3. COR.

COR. 6. If the right line joining the vertices of two diameters of a conic ſection be ordinately applied to an axis, the right lines touching the ſection in theſe vertices ſhall meet in an axis, make equal angles with an axis, and be equal between themſelves; moreover, the parameters of the diameters, the ordinates applied from each vertex to the alternate diameter, the abſciſſes of the diameters intercepted thereby, and in the caſe of the ellipſe and hyperbola,

S the

FIG. the diameters also and the diameters conjugate to them shall be equal between themselves.

168, 169, 170. Let the right line AD joining the vertices A, D, of two diameters Aa, Dd, of a conic section be ordinately applied to an axis, then AI, DI, touching the section in A, D, will meet in the axis to which AD is ordinately applied, and be equal between themselves, and make equal angles with an axis (3. Cor.). Wherefore the parameters of the diameters (23.), and in the case of the ellipse and hyperbola the conjugate diameters (4. Cor. 23.), as also the diameters shall be equal between themselves (6. Cor. 23.). But farther, if DO, AE ordinately applid to Aa, Dd be drawn, and C be the diameter of the section; I say, that DO shall be equal to AE, and AO to DE. For because the angles IDA, IAD are equal between themselves, the alternate angles DAE, ADO, shall also be equal between themselves. And in the ellipse and hyperbola, because DC is equal to AC, the angle ADC is equal to the angle DAC, while in the parabola the diameters being parallel to the axis, the angles ADE, DAO are right and therefore equal. Wherefore in each section the triangles ADO, DAE are equiangular, and the homologous side AD is common to each, therefore the other homologous sides DO, AE, as also AO, DE, shall be equal between themselves (26. e. 1.).

PROP. 28.

If through the focus of a conic section, right lines be drawn, which are intercepted by the same section; that is the least, which is ordinately applied to the focal axis, and that which is nearer to the axis is greater than that which is more remote, and

In the case of the ellipse, the focal axis is the greatest of all. 6

But

But in the oppoſite hyperbolas, the leaſt focal right line is the axis, and of others, that which is nearer to the axis is leſs than that which is more remote. FIG.

Let F be the focus, XX the directrix, and AB the axis of a conic ſection; I ſay, that of all the right lines drawn through F, and intercepted by the ſection, the leaſt is DFd, which is ordinately applied to the axis AB. 133.

Let EFe be any other focal line meeting the ſection in E, e. Ee is not biſected in F, becauſe not ordinately applied to AB. Biſect therefore Ee in O, and draw EI, ei, OK, DH, perpendicular to the directrix. Becauſe Ee is not parallel to the directrix, one of its ſegments FE, will decline from the directrix, the other Fe incline towards the directrix, and EI will be greater than ei. But EI is to EF as ei is to eF (1. COR. 1.), therefore EF is greater than eF, and the point O will be on the part of F towards E, viz. the point O is remoter from the directrix than F, and OK is greater than FL or DH. But OE or Oe being equal to the ſemidiameter of the generating circle deſcribed round O (2. COR. DEF.), OK is to OE as DH is to DF (1. COR. DEF.), and therefore OE will be greater than DF. Wherefore the double Ee will be greater than the double Dd.

Again, of two focal right lines Ee, Gg, intercepted by the ſection, I ſay, that Gg, which is nearer to the axis AB, is greater than Ee, which is more remote.—Biſect Gg in o, and draw ok perpendicular to the directrix. Then for the ſame reaſon as above, the point o will be remoter from the directrix than O, and ok will be greater than OK. And by the ſame reaſoning it is ſhewn that ok is to Go (as DH is to DF, viz.), as OK is to EO. Wherefore Go will be greater than EO, and the double Gg be greater than the double Ee.

I ſay alſo, that in the caſe of the ellipſe, the axis AB is the greateſt line, intercepted by the ſection, which can be drawn through the focus F. Biſect AB in the centre C. Becauſe the

FIG. vertex A is that point in the ellipse, which is most remote from the directrix, therefore by the same reasoning as above, it is shewn that C is remoter from the directrix than o, and that CL is greater than oK. Also for the same reason, CL is to CA as oK is to Go, and therefore CA is greater than Go, and the double AB greater than the double Gg.

134. II. In the opposite hyperbolas, I say, that of all the right lines drawn through the focus F, and inscribed between the sections, the axis AB is the least, and that Ee remoter from the axis is greater than Gg, which is nearer to the axis.

Let AB, Ee, Gg, meet the directrix in L, H, K, and draw EI, GN, perpendicular to the directrix. Because GK is greater than GN (19. e. 1.), and GN is greater than BL (2. Cor. 6.), therefore GK is much greater than BL. For the same reason is gK much greater than AL. Wherefore the whole Gg is much greater than the whole AB.

I say also, that Ee is greater than Gg. Draw EO parallel to Gg, meeting the directrix in O. Because FG falls within the angle BFE, EO will fall within the alternate angle FEI, and EH be greater than EO. But on account of the equi-angular triangles EIO, GNK, EI is to GN as EO is to GK, therefore EI being greater than GN (2. Cor. 6.), EO will be greater than GK, and consequently EH will be much greater than GK. For the same reason is eH much greater than gK, therefore the whole Ee is much greater than the whole Gg.

Cor. 1. If two right lines touching the same section or each an opposite hyperbola, meet each other, that shall be the greater, which is nearer to the focal axis.

133. Let the right lines PQ, VQ, meeting each other in Q, touch the same section, or each an opposite hyperbola in P, V; then if VQ be inclined to the focal axis AB in a less angle than PQ is, I say, that VQ is greater than PQ.

Through the focus F draw Gg, Ee, parallel to VQ, PQ, and meeting the section in G, g, E, e. Because VQ, PQ, touch the section,

ſection, they are neither parallel to an aſſymptote nor to a tranſverſe diameter of the hyperbola, nor to the diameters of a parabola (II.), and therefore G g, E e, which are parallel to them, will meet the ſame ſection in the points G g, E e (8.). Wherefore by this PROP. G g will be greater than E e. But G g is to E e in the duplicate proportion of V Q to P Q (23.), therefore alſo V Q will be greater than P Q.

COR. 2. For the ſame reaſon, if a right line touching a conic ſection meet a right line cutting the ſame ſection or the oppoſite hyperbola, then accordingly as one of theſe lines is nearer to the focal axis than the other, the ſquare of the touching line, or the rectangle under the ſegments of the cutting line, between the ſection and the concourſe, ſhall be the greater.

COR. 3. For the ſame reaſon, if two right lines, which each cut the ſame ſection or an oppoſite hyperbola, meet each other, the rectangle under the ſegments of the one, which is nearer to the focal axis, is greater than the rectangle under the ſegments of the other.

COR. 4. And in like manner, if two right lines inſcribed in the ſame ſection, or in the oppoſite hyperbolas, meet each other, the rectangle under the ſegments of the one which is remoter from the focal axis, ſhall be greater than the rectangle under the ſegments of the other.

COR. 5. In the ellipſe, the greateſt diameter is the axis tranſverſe, which is therefore called the AXIS MAJOR; the leaſt is the axis ſecundus, which is therefore called the AXIS MINOR; and that diameter which is nearer to the axis tranſverſe is greater than that which is more remote.

Let A B be the axis tranſverſe, T t the axis ſecundus, and S s, R r, two other diameters, of which S s is nearer to A B than R r; I ſay, that A B is the greateſt, T t the leaſt, and that S s is greater than R r.

Draw through the focus F the focal lines D d, G g, E e, parallel to T t, S s, R r, and meeting the ellipſe in D, d, G, g, E, e.

Then

FIG. Then AB is greater than Gg, by the PROP, but AB, Ss, meeting each other in C, the rectangle ACB is to the rectangle SCs as AB is to Gg (23.). Therefore the rectangle ACB is greater than the rectangle SCs, that is, because AB, Ss are bisected in C, the square of AC is greater than the square of SC, and AC is greater than SC, and the double AB is greater than the double Ss.

By the same reasoning it is proved that any diameter Ss is greater than the axis secundus Tt; and also that Ss, which is nearer to the transverse axis, is greater than Rr, which is more remote.

COR. 6. In the opposite hyperbolas, the axis transverse is the least diameter, and that diameter, which is remoter from the axis transverse, is greater than one which is nearer to it.

This is proved in the same manner as the last corollary.

PROP. 29.

If a right line touching a conic section in the vertex of the focal axis, meet a right line touching the section in any other point, and from this other point be drawn an ordinate to the axis, the segment of the line, which touches the section in the vertex, intercepted between the vertex and the concourse, shall be equal to, less, or greater than, the abscis of the axis, intercepted between the vertex and the ordinate, accordingly as the ordinate passes through the focus, is remoter from, or nearer to, the vertex than the focus is.

138. I. Let DF ordinately applied to the focal axis of a conic section pass through the focus F, and meet the section in D, and, A being

ing the vertex of the focal axis AF; let AB, DB, touching the section in A, D, meet each other in B. I say, that AB is equal to AF. FIG.

Let the axis meet the directrix in E, and draw DH parallel to the axis, meeting the directrix in H. The tangent DB passes through E (6. COR. 9.), and DH or EF is to FD as AE is to AF (1. COR. 1.) But on account of the parallels, EF is to FD as AE is to AB; therefore AB is equal to AF.

II. If DF ordinately applied to the axis, meet it in the point F, 138.
remoter from the vertex A than the focus F is, and DB touching the section in D, meet AB in B; I say, that AB is less than AF. ——The same things remaining, draw BG, BG, parallel to the axis, meeting DF, DF, in G, G, and BI parallel to BD, meeting DF in I. Also let the focal ordinates parallel to AB, BD, BD, be represented by X, Y, Z. Because BD is inclined to the axis in a greater angle than BD, Y is remoter from the axis than Z, and therefore will be less than Z (28.). Wherefore X has to Y a greater ratio than X has to Z (8. e. 5.). But X is to Y in the duplicate ratio of AB to BD, and X is to Z in the duplicate ratio of AB to BD (23.); therefore AB has to BD a greater ratio than AB has to BD. Again by CASE 1., AB is equal to AF or BG, and on account of the equiangular triangles BGD, BGI, BG or AB is to BD as BG is to BI; therefore BG has to BI a greater ratio than AB has to BD. But because BD falls within the angle GBI, it will be less than BI, and BG will have to BD a greater ratio than BG has to BI (8. e. 5.); therefore BG will have to BD a much greater ratio than AB has to BD. Wherefore AB is less than BG or AF.

III. Let the point F be nearer to the vertex A than the focus F; 137.
I say, that AB is greater AF.

Other things remaining the same, because now BD makes a less angle with the directrix than BD, therefore by a parity of reasoning, AB will have to BD a less ratio than AB has to BD, and because AB or BG is to BD as BG is to BI, therefore BG will have

to

FIG. to BI a less ratio than Ab has to BD. But because BD now falls without the angle GBI, BD will be greater than BI, and BG will have to BD a less ratio than BG has to BI. Wherefore BG has to BD a much less ratio than Ab has to BD, and consequently Ab is greater than BG, viz. than Af.

Schol. If the right line be drawn, touching a conic section in the vertex of the focal right line, which is ordinately applied to the focal axis, and a parallel to the ordinates applied to the same axis meet the section, the axis and also the touching line; the segment of the parallel between the axis and the tangent shall be equal to the distance between the focus and the point in which the parallel meets the section.

121. Through the focus F of a conic section let AB, ordinately applied to the focal axis FQ, meet the section in A; and join A and Q the concourse of the axis with the directrix. AQ will touch the section in A (6. Cor. 9.). Draw any parallel to AB, meeting the section in I, AQ in L, and FQ in H, and join FI; I say, that HL is equal to FI.

By the 1. Cor. 1. QH is to FI, (as QF is to FA, viz.) as QH is to HL. Therefore HL is equal to FI.

This Scholium is demonstrated in the same manner as the 1. Case of this Prop., and is that more general property, which comprehends the first Case.

PROP. 30.

If a right line touch a conic section, and from the point of contact two right lines be drawn to meet the focal axis, the one ordinately applied to the axis, the other perpendicular to the touching line; the distance between the focus and the point of contact shall be a mean proportional between the portions of the axis, intercepted

intercepted between the focus and the perpendicular, the ordinate and the directrix. FIG.

Also, the same two portions of the axis are to each other in the duplicate ratio of the principal semi-parameter to the distance of the focus from the directrix.

Let AD be a conic section, F the focus, XX the directrix, 139, 140.
FA the focal axis, meeting the section in A and the directrix in E. 141.
If a right line DH touch the section in D, and DP perpendicular to DH, DG ordinately applied to the focal axis, meet the axis in P, G; I say, that FD, being joined, shall be a mean proportional between FP, GE.

Let DH meet the directrix in H, and drawing DI perpendicular to the directrix, join FI, FH. Because the angle DFH (1. Cor. 9.), and also the angle DIH, are right, the four points D, F, H, I, are in a circle, and the angle DFI is equal to the angle DHI. Again, because the angles PDH, PEH, are each right, the four points D, P, H, E, are also in a circle, and the angle DPF is equal to the angle DHI. Therefore the angle DFI is equal to the angle DPF. But the angle DFP is equal to the alternate angle FDI; therefore the triangles PDF, FID, are equiangular, and FP is to FD as FD is to DI. But DI is equal to GE, therefore FP is to FD as FD is to GE.

I say also, that FP is to GE in the duplicate proportion of the principal semi-parameter to FE.—For, because FD is a mean proportional between FP, DI, therefore FP is to DI or GE in the duplicate ratio (of FD to DI, viz.) of the principal semi-parameter to FE (1.).

Cor. 1. In the parabola, the subnormal PG is equal to the semi-latus rectum of the axis.

Because in the parabola, the principal semi-parameter is equal 139.
to FE (2. Def.), and FP is to GE in the duplicate ratio of this

T semi-

FIG. ſemi-parameter to FE, therefore FP is equal to GE, and adding or taking away FG, which is common, PG will be equal to FE.

140, 141. COR. 2. In the ellipſe and hyperbola, the ſubnormal PG is to FE the diſtance of the focus from the directrix, as CP, the diſtance of the centre of either ſection from the point P is to CF the diſtance of the centre from the focus.

FP is to GE in the duplicate proportion (of the principal ſemiparameter to FE, viz.) of AC to CE (4. COR. DEF.). But becauſe AC is a mean proportional between CF, CE (17. COR. DEF.), therefore CF is to CE in the duplicate proportion of AC to CE, and conſequently, ex æquo, FP is to GE as CF is to CE, or invertendo, GE is to FP as CE is to CF. Wherefore, dividendo or componendo, CG is to CP as CE is to CF; and again, dividendo or componendo, PG is to FE as CP is to CF.

NOTE. Theſe two Corollaries agree; for in the parabola, the centre being at an infinite diſtance or vaniſhing, CP, CF, become equal, and therefore PG, FE, alſo.

COR. 3. In the ellipſe and hyperbola, the ſubnormal, viz. that part of either axis, which is intercepted between the ordinate and the perpendicular, is to the portion of the axis intercepted between the centre and the ordinate, as the ſemi-latus rectum of the ſame axis is to the ſemi-axis.

140, 141. Firſt, if the perpendicular meet AB the tranſverſe axis, and L repreſent its ſemi-latus rectum; I ſay, that PG is to CG as L is to AC.

By the preceding Corollary, PG is to CP as FE is to CF, therefore, componendo or dividendo, PG is to CG as FE is to CE. But FE is to L as CE is to AC (4. COR. DEF.); therefore, alternando and ex æquo, PG is to CG as L is to AC.

Secondly, if the perpendicular meet ab the axis ſecundus in p, and l repreſent its ſemi-latus rectum. Draw Dg perpendicular to ab, then pg is the ſubnormal, and I ſay, that pg is to Cg as l is to aC.

Becauſe

Because of the parallels, PG is to CG (as CG is to PG, viz.), as AC is to L. But AC is to L as L is to AC (24. COR. DEF.). Therefore, ex æquo, PG is to CG as L is to AC. FIG.

PROP. 31.

If two right lines touching a conic section or the opposite hyperbolas meet each other, and a right line parallel to one of them, and cutting the section or either hyperbola, meet the other touching-line and also the right line joining the two points of contact; the segment of the parallel, intercepted between the other tangent and the right line joining the points of contact, shall be a mean proportional between the segments of the parallel intercepted between the tangent and the section.

But if the parallel also touch the section or an opposite hyperbola, the segment intercepted between the tangent and the right line joining the points of contact shall be equal to the segment intercepted between the tangent and section.

Let the right lines AS, BS, touching a conic section or an opposite hyperbola in A, B, meet in S, and DE parallel to BS meet the section or an opposite hyperbola in D, E, AS in G, and AB in H; I say, that GH shall be a mean proportional between DG, GE. 76.

The square of AG is to the rectangle DGE (as the square of AS is to the square of BS (2. COR. 23.), viz. on account of the parallels,) as the square of AG is to the square of GH. Where-

FIG. fore the ſquare of GH is equal to the rectangle DGE, and GH is a mean proportional between DG, GE.

77. But if GD touch the ſection or either hyperbola in D, then GH ſhall be equal to GD.

For the ſquare of AG is to the ſquare of GD (as the ſquare of AS is to the ſquare of BS (1. COR. 23.) viz.) as the ſquare of AG is to the ſquare of GH. Wherefore GH is equal to GD.

PROP. 32.

175, 176. 177, 178. 179. **If a right line AS touching a conic ſection, be parallel to a diameter Bb, or meet it in S, and from the point of contact A be drawn AD ordinately applied to the ſame diameter Bb, and meet it in D, alſo EG be drawn parallel to AD, meeting the ſection in E, G, and AS in I; then,**

175, 176. 177. **If Bb be any diameter of the ellipſe or parabola, or a tranſverſe diameter of the hyperbola, and at the vertex B be drawn BH parallel to AD, and meet AS in H; I ſay, that the rectangle EIG is to the ſquare of LD as the ſquare of BH is to the ſquare of BD.**

178, 179. **But if Bb be a diameter ſecunda of the hyperbola, let AS meet CP the tranſverſe diameter conjugate to Bb, in the point H; I ſay, that the rectangle EIG is to the ſquare of LD as the ſquare of CH is to the ſquare of CB the ſemidiameter ſecunda.**

175, 176. 177. CASE 1. When Bb is any diameter of the parabola and ellipſe, or a tranſverſe diameter of the hyperbola.

Becauſe

FIG.

Because B is a vertex of the diameter Bb, and BH is parallel to the ordinates applied to Bb, it will touch the section in B; wherefore because the right line AS touching the section in A meets the parallels EG, BH, in I, H, the rectangle EIG is to the square of BH (as the square of AI is to the square of AH (1. COR. 23.), viz.) as the square of LD is to the square of BD; and, alternando, the rectangle EIG is to the square of LD as the square of BH is to the square of BD.

CASE 2. When Bb is a diameter secunda of the hyperbola. 178, 179.

If AS be parallel to the diameter secunda Bb, the point H falls in A, and CH becomes the same with AC. But the rectangle EIG is to the square of AI (as the focal line parallel to EG or AC is to the focal line parallel to AL or CB (23.), viz.) as the square of AC or HC is to the square of CB (4. COR. 23.). But AI is equal to LC or LD, therefore the rectangle EIG is to the square of LD as the square of HC is to the square of CB.

If AS be not parallel to Bb, but meet it in S; draw AO parallel to Bb, meeting CP in O. The square of CP is equal to the rectangle OCH (9. COR. 24.), and taking away the common square of CH, the remaining rectangle PHp will be equal to the remaining rectangle OHC (3. e. 2. & 5. e. 2.). But also, because CB is a mean proportional between SC, CD (9. COR. 24.), the square of CB is to the square of CD (as SC is to CD, viz. as HC is to OH, viz.) as the square of CH is to the rectangle OHC or PHp. Again, because AS touching the section in A meets the parallels EG, Pp, which cut the sections, the rectangle EIG is to the rectangle PHp (as the square of AI is to the square of AH (1. COR. 23.), viz.) as the square of LD is to the square of CD. But it has been shewn, that the square of CH is to the rectangle PHp as the square of CB is to the square of CD. Therefore, ex æquo, the rectangle EIG is to the square of ED as the square of CH is to the square of CB.

FIG. 180, 181. COR. 1. If the diameter Bb be the focal axis, and D be the focus of the section; or when Bb is the axis secundus of the hyperbola, if HC be equal to CB, the rectangle EIG shall be equal to the square of LD.

For in the first case, BH being equal to BD (29.), and in the second case, HC being equal to CB, it follows directly from the proposition, that the rectangle EIG is equal to the square of LD.

180. COR. 2. If the diameter Bb be the focal axis, and D be the focus, and EG meet Bb in L, DE being joined shall be equal to IL.

For the rectangle EIG being, by the preceding COR., equal to the square of LD, add the common square of EL, and the square of IL will be equal to the square of DE, viz. IL will be equal to DE.—This COR. has already appeared in SCHOL. 29.

181. COR. 3. If the diameter Bb be the axis secundus of the hyperbola, meeting EG in L, and CH be equal to CB, then DI, being joined, shall be equal to EL.

For the rectangle EIG is then equal to the square of LD, therefore adding the common square of IL, the square of EL will be equal to the square of DI, and EL be equal to DI.

PROP. 33.

182, 183, 184, 185. **The same things remaining, if Bb be the focal axis, and the point D be remoter from the vertex B than the focus is, in which case BH will be less than BD (29.), and, if round the vertex B with the distance BH a circle be described, meeting BH again in h, and DQ be drawn to touch this circle in Q and meet HB in T, also through T be drawn TS, verging to the concourse S, and meeting EG in K and**

DK

DK be joined; I say, that the rectangle EKG is to the square of DK as the square of BH is to the square of BD. FIG.

Draw *h*S, verging to the concourse S, let it meet EG in *i*, and join BQ which will be perpendicular to DT. The square of BQ is to the square of BD as the square of TQ is to the square of BT (8. e. 6.), that is, because BQ is equal to BH, and the square of TQ is equal to the rectangle HT*h*, the square of BH is to the square of BD (as the rectangle HT*h* is to the square of BT viz.) as the rectangle IK*i* is to the square of KL. But also the rectangle EIG is to the square of LD as the square of BH is to the square of BD (32.), therefore, ex æquo, the rectangle IK*i* is to the square of KL as the rectangle EIG is to the square of LD. Wherefore, componendo, * the rectangle EKG is to the square of DK (as the rectangle EIG is to the square of LD, viz.) as the square of BH is to the square of BD.

COR. The rectangle EKG is to the square of DE as the square of BT is to the square of BD.

For the rectangle EKG is to the square of DK as the square of BH or BQ is to the square of BD; therefore, dividendo, † the rectangle EKG is to the square of DE (as the square of BQ is to the square of QD, viz.) as the square of BT is to the square of BD.

* This composition supposes that the rectangles IK*i*, EIG together are equal to the rectangle EKG. Because I*i*, EG, are each bisected in L, the square of KL is equal to the rectangle IK*i* together with the square of IL, and also to the rectangle EKG together with the square of EL (6. e. 2.). Therefore the rectangle IK*i* together with the square of IL is equal to the rectangle EKG together with the square of EL. Take away the common square of EL, and the rectangle IK*i* together with the rectangle EIG is equal to the rectangle EKG (6. e. 2.).

† This division supposes that the square of DK is equal to the rectangle EKG together with the square of DE.—The square of KL is equal to the rectangle EKG together with the square of EL. Add the common square of LD, and the square of DK will be equal to the rectangle EKG together with the square of DE (47. e. 1.).

PROP.

FIG.

PROP. 34.

186, 187. The same things remaining, if AS be parallel to the axis secundus Bb of an hyperbola, or meet it in S, and also meet the axis transverse Pp in H; then joining H, and B a vertex of Bb, if C be the centre of the hyperbolas, and in CP on the parts of C opposite to H, be taken CT a fourth proportional to BH, CB, CH, and TS, meeting EG in K, be drawn, verging to the concourse S, also DK be joined; I say, that the rectangle EKG shall be to the square of DK as the square of CH is to the square of BC.

In CP on the parts of C opposite to H take Ch equal to CH, and verging to S draw Sh, meeting EG in i. Because BH, BC, CH, CT, are proportionals, and CB is less than BH, CT will be less than CH, and the point T will be between the terms H, h, as also the point K be between the terms I, i. Then the square of BH is to the square of BC as the square of CH is to the square of CT; therefore, dividendo, the square of CH will be to the square of BC (as the rectangle HTh is to the square of CT, viz.) as the rectangle IKi is to the square of KL. But also the square of CH is to the square of BC as the rectangle EIG is to the square of LD (32.); therefore, ex æquo, the rectangle IKi is to the square of KL as the rectangle EIG is to the square of LD; and, componendo, * the rectangle EKG is to the square of DK (as the rectangle EIG is to the square of LD, viz.) as the square of CH is to the square of BC.

Cor. The rectangle EKG is to the square of DE as the square of CT is to the square of BC.

For the rectangle EKG is to the square of DK as the square of CH is to the square of BC; whence, componendo, † the rectangle EKG is to the square of DE (as the square of CH is to the square of BH, viz.) as the square of CT is to the square of BC.

* Vide 1. of the preceding notes.

† This is illustrated in the same manner as the second of the preceding notes.

PROP.

FIG.

PROP. 35.

The ſame things remaining as in PROP. 33.; if a right line parallel to AD meet the ſection in the point E, and TS in K; I ſay that DE, DK, being joined, ſhall be in the conſtant ratio of BH, BT.

For every thing remaining the ſame, the rectangle EKG is to 182, 183.
the ſquare of DK as the ſquare of BH is to the ſquare of BD 184, 185.
(33.), and alſo, the rectangle EKG is to the ſquare of DE as the the ſquare of BT is to the ſquare of BD (COR. 33.). Therefore, ex æquo perturbate, the ſquare of DE is to the ſquare of DK as the ſquare of BH is to the ſquare of BT, and conſequently DE is to DK as BH is to BT.

PROP. 36.

The ſame things remaining as in PROP. 34., if a right line parallel to AD meet the hyperbola in E, and TS in K; I ſay, that DE, DK, being joined, ſhall be to each other in the conſtant ratio of CH to CT.

For, every thing remaining the ſame, the rectangle EKG is to 186, 187.
the ſquare of DK as the ſquare of CH is to the ſquare of BC (34.), and alſo, the rectangle EKG is to the ſquare of DE as the ſquare of CT is to the ſquare of BC (COR. 34.). Therefore, ex æquo perturbate, the ſquare of DE is to the ſquare of DK as the ſquare of CH is to the ſquare of CT, and conſequently DE is to DK as CH is to CT.

PROP. 37.

PART I. If two right lines touching a conic ſection or the oppoſite hyperbolas, meet each other, or a right line touching an hyperbola meet an aſſymptote; a right line drawn through the concourſe to meet the

FIG. section or sections in two points, shall be harmonically divided in the concourse, in its concourses with the section, and in its concourse with the right line joining the points of contact, or in the particular case of the hyperbola, in its concourse with a parallel to the assymptote drawn through the single point of contact.

But if the right line drawn be parallel to a diameter of the parabola or to an assymptote of an hyperbola, and therefore meet the section in one point only, (6.), the right line drawn, viz. intercepted between the concourse of the tangents and the right line joining the points of contact, shall be bisected in the section.

Also in the case when a single tangent to the hyperbola meets an assymptote, a right line drawn through the concourse parallel to the other assymptote, and meeting the parallel drawn through the point of contact, shall be bisected in the section.

Part II. If through the concourse of the tangents two right lines be drawn, one, the diameter of the section, the other parallel to the ordinates applied to the diameter; a right line drawn through the concourse of the diameter with the right line joining the points of contact, and meeting the section or sections again in two points, shall be harmonically divided therein, in the point through which it is drawn, and in its concourse with the parallel.

But if the line drawn be parallel to an assymptote of the hyperbola, the segment intercepted between the parallel and the point through which it is drawn, shall be bisected in the section.

PART I. CASE 1. Let AQ touching a conic ſection in A, meet BQ touching the ſame ſection or the oppoſite hyperbola in B, or meet an aſſymptote CK of the hyperbola in Q. If AB be joined in the firſt inſtance, or in the latter AB be drawn parallel to CK, and through Q be drawn a right line meeting the ſection or ſections in D, E, and AB in R; I ſay, that QR ſhall be harmonically divided in D, E. FIG. 111, 113.

Firſt, if AQ touching the ſection meet BQ, alſo touching the ſection in Q: draw the diameter of the ſection through Q meeting AB in O, and round O deſcribe the generating circle. Draw FP reſpondent to AB, meeting the circle in the points A, B, reſpondent to A, B, in the ſection (1. COR. 3.), alſo draw AQ, BQ, DE, correſpondent to AQ, BQ, DE, the latter meeting the circle in D, E, reſpondent to the points D, E, in the ſection (1. COR. 3.), and meeting AB in R. From F the focus of the ſection draw FD, FE, FR, FQ, and join alſo OD, OE, OR, OQ. Becauſe AQ, BQ, DE, meet in Q, their focal correſpondents AQ, BQ, DE, will meet in one common point Q, correſpondent to Q (4.); and alſo, becauſe AQ, BQ, touch the ſection in A, B, their focal correſpondents AQ, BQ, will touch the circle in the reſpondent points A, B (9.). Wherefore QR will be harmonically divided in D, E (LEM. 9.), and OD, OE, OR, OQ, will be harmonicals. But becauſe D, E, in the ſection, are reſpondent to the points D, E, in the circle, FD, FE, will be parallel to OD, OE, (19. DEF.), and becauſe R, Q, are correſpondent to R, Q, (4.), therefore for the ſame reaſon (20. DEF.), are FR, FQ, parallel to OR, OQ. Wherefore FD, FE, FR, FQ, are alſo harmonicals, and conſequently QR is harmonically divided in D, E, (1. COR. LEM. 4.). 111.

If AB paſs through the focus F, the tangents AQ, BQ, meet in the directrix, and FQ is perpendicular to AB (5. COR. 9.). Wherefore AQ, BQ, DE, FQ, meeting all in the directrix, their focal correſpondents AQ, BQ, DE, OQ, will all be parallel to each other (14. COR. DEF.), and all be perpendicular to AB, becauſe all 112.

 parallel

FIG. parallel to FQ, the common focal respondent to AQ, BQ, DE. Therefore DE is bisected in R (3. e. 3.), and OD, OE, OR, OQ, are harmonicals (3. COR. LEM. 4.), and as before, FD, FE, FR, FQ, being parallel to them, are also harmonicals, and QR is harmonically divided in D, E.

113. But if AQ touching an hyperbola in A meet an assymptote CK in Q: first, let AB not pass through the focus F. To K the concourse of the assymptote with the directrix draw FK, AK, and join AF. Because FK is perpendicular to CK (19. COR. DEF.), and therefore to AB, it is not perpendicular to AF, and consequently AK does not touch the hyperbola in A (9.), viz. AQ does not coincide with AK, but meets the assymptote in some other point than K. Round A describe the generating circle, which will pass through H the concourse of AB with the directrix, and also through the focus F (8. & 5. COR. DEF.). Draw FP respondent to AQ, and HQ parallel to FK. Because AH, HQ, are respectively parallel to CK, FK, they will be the correspondents to CK, FK (18. DEF.), wherefore the conic lines CQ, AQ, meeting in a point Q, which is not in the directrix, their focal correspondents FP, HQ, will meet in a correspondent point Q (4.), and for the same reason DE the focal correspondent to DE will pass through Q. Because AQ touches the section, its focal respondent FP will touch the circle in the point F, the point wherein it meets it (9.), viz. in this case the points A, F, coincide. Also, because CK, FK, are at right angles to each other (19. COR. DEF.), the right lines AH, HQ, being parallel to them, will be at right angles to each other, viz. HQ also touches the circle in H. Wherefore joining HF to meet DE in R, and every thing being drawn as in the preceding case, QR will be harmonically divided in D, E, and AD, AE, AR, AQ, will be harmonicals, and therefore FD, FE, FR, FQ, being parallel to them, will also be harmonicals, and QR be harmonically divided in D, E.

114. But if AB pass through the focus F, then FK, being perpendicular to CK, is perpendicular to AF, and therefore AK touches the

the section in A (1. COR. 9.), that is, AQ in this case meets the assymptote in K, or the points Q, K, coincide. Wherefore because the three conic right lines AQ, CQ, DE, meet in the directrix, their focal respondents or correspondents FP, Hq, de, will be parallel to each other (14. COR. DEF.); and FP, Hq, being perpendicular to AF or AB, de will be perpendicular thereto, and therefore will be bisected by AB in the point r (3. e. 3.). Therefore drawing Aq parallel to FP or FQ, the four right lines Ad, Ae, Ar, Aq are harmonicals (3. COR. LEM. 4.), and consequently, as before, the four right lines FD, FE, FR, FQ, which are parallel to them, are also harmonicals, and QR is harmonically divided in D, E. FIG.

CASE 2. If QR drawn from the concourse of the tangents be parallel to the diameters of a parabola, or to an assymptote of an hyperbola, and therefore meet the section in one point D only; I say, that QR shall be bisected in D. 115, 116.

In this case, as applied to the parabola, the point E vanishes, QR being a diameter, the point R coincides with O, and de respondent to QR passes through the point e, in which the generating circle touches the directrix. But in the hyperbola, because QR is parallel to an assymptote, Oe parallel thereto will meet the directrix in the point e, one of the points, in which the generating circle cuts the directrix, (8. COR. DEF.), and therefore de correspondent to QR passes through e. In the same manner therefore, as before, it is shewn Od, Oe, Or, Oq, are harmonicals, and if FE be drawn parallel to Oe, or QR, that FD, FE, FR, FQ, are harmonicals. But because QR parallel to one of these four harmonicals, viz. to FE, falls upon the other three, it will be bisected in the middle concourse (LEM. 4.); viz. QD is equal to DR. 115. 116.

But in this case, if a single tangent AQ meet an assymptote CK of the hyperbola in Q, and QR be drawn parallel to the other assymptote, 117.

FIG. ſymptote, meeting AB in R, and the hyperbola in the ſingle point D (6.); then alſo, ſhall QR be biſected in D.

Firſt, if AB do not paſs through the focus, then for the ſame reaſons, as above, AQ will not meet the aſſymptote in the point K. Wherefore every thing remaining the ſame, as in the ſimilar caſe above, Ad, Ae, Ar, Aq, will be harmonicals, and FE being drawn parallel to Ae or QR, the four right lines FD, FE, FR, FQ, being parallel to the former, will alſo be harmonicals. Wherefore QR, being parallel to one of theſe four harmonicals, viz. to FE, will be biſected in the middle concourſe D (Lem. 4.).

118. Secondly, if AB paſs through the focus F, for the ſame reaſons as in the caſe, which anſwers to it above, it will appear that FQ, Hq, de, Aq, are each parallel to each other, and each perpendicular to AB; that Ad, Ae, Ar, Aq, are harmonicals; and therefore drawing FE parallel to Ae or QR, that FD, FE, FR, FQ, are alſo harmonicals, and conſequently that QR being parallel to FE, and falling upon the three other harmonicals FD, FR, FQ, will be biſected in the middle concourſe D.

Part II. Case 1. Other things remaining the ſame, and the diameter QO being drawn to meet in O the right line AB, which joins the points of contact, if through Q the concourſe of the tangents be drawn a right line QR parallel to AB; I ſay, that every right line drawn through O to meet the ſection or ſections in D, E, and QR in R, ſhall be harmonically divided in O, D, E, R.

120. Firſt, let AB meet the directrix in P, and every thing in this part, except what is peculiar to it, remaining the ſame as in the firſt part, becauſe AB paſſing through the centre O of the generating circle is parallel to QR, the focal correſpondent to QR will paſs through P (13. Cor. Def.). Let this correſpondent, viz. qP be drawn. Becauſe the three conic lines AQ, BQ, RQ, meet in Q, their focal correſpondents Aq, Bq, Pq, will meet in a correſpondent point q (4.). But Aq, Bq, touch the circle in A, B, becauſe

 they

they are correspondent to AQ, BQ, which touch the section in the points A, B, respondent to A, B, in the circle (9.), and because the centre O of the circle is within the section, the focus F shall be within the circle (2. COR. 5.). Wherefore QF will be harmonically divided in its concourses with the circle (LEM. 9.). Also, because round O the concourse of the diameter QO with the right line AB ordinately applied to it (18.), the generating circle is described, FP the focal respondent to AB shall be harmonically divided in A, B, its concourses with the circle (14.) Wherefore every right line drawn through F to cut the circle and meet QP will be harmonically divided (1. COR. LEM. 10.); viz. FR, which is so drawn will be harmonically divided in D, E. Join FE, OE, which are parallel to each other, because E, E, are respondent points; as for the same reason are FD, OD, and FR, OR, being joined, will also be parallel to each other (4.), because the points R, R, are correspondent. FR therefore being harmonically divided in D, E, the lines OD, OE, OR, OF, will be harmonicals, and consequently FD, FE, FR, FO, which are respectively parallel to them, are also harmonicals, and OR is harmonically divided in D, E. FIG.

Secondly, when AB is parallel to the directrix, the demonstration is only rendered shorter, for then QR, QR, AB, become also parallel to the directrix, and parallel to each other. Wherefore FR being harmonically divided in D, E (LEM. 9. & 10.), the demonstration proceeds as before. 119

If AB pass through the focus F, then as in the like instance in the first part, the points A, B, and their respondent points A, B, coincide; the point Q falls in the directrix, and AQ, BQ, FQ, PQ, or RQ become parallel to each other, and perpendicular to the directrix. Also, FP being harmonically divided in A, B (3. COR. 1.), every right line drawn through F to cut the circle and meet PQ will be harmonically divided (LEM. 10.). Wherefore, as in the preceding, FR is harmonically divided in D, E, whence the demonstration proceeds as before. 123.

When

FIG. When in this latter case AB is parallel to the directrix, the point
121. O falls in the focus F, the diameter QO becomes the focal axis, QR the directrix itself, and therefore the proposition is the same as the 3. Cor. 1.

122. Case 2. When in the case of the hyperbola, OR is drawn parallel to an assymptote Ck; I say, that OR shall be bisected in D, its single concourse with the section.

Because OR, being parallel to an assymptote, meets the directrix in e, one of the points in which the circle cuts the directrix, therefore de the respondent to OR passes through e, and its other concourse with the circle is d, the respondent to the single point D. But for the same reason as in the other cases, Fr is harmonically divided in d, e; wherefore Od, Oe, Or, OF are harmonicals, and consequently, if FE be drawn parallel to Oe, the four right lines FD, FE, FR, FO, will also be harmonicals. But OR being the same right line with Oe, is parallel to FE, and meeting the other three harmonicals FD, FR, FO, it will be bisected in the middle concourse D (Lem. 4,).

Cor. 1. If a right line, meeting a conic section or the opposite hyperbolas in two points, be harmonically divided; or if in a diameter of the parabola two points equidistant from the vertex be assumed, and from the harmonic or equidistant point within the section, an ordinate be applied to the diameter drawn through the other point, the right lines joining the vertices of the ordinate and the harmonic or equidistant point without the section shall touch the section or sections.

111, 115. Let DE meeting a conic section or the opposite hyperbolas in D, E, be harmonically divided in Q, R, of which the point Q without the section is not in an assymptote of the hyperbola; or if in QD a diameter of the parabola two points Q, R, equidistant from the vertex D be assumed; and ARB be ordinately applied to the diameter drawn through Q: I say, that QA, QB, being drawn, will touch the section or sections.

If

If not, let the right lines drawn from Q to touch the ſection or ſections be drawn. The right line joining the points of contact will be different from AB, and meet DE in another point than R, and the other point will alſo be harmonic to the ſame point Q, or towards the ſame parts another ſegment than RD will be intercepted equal to DQ, which is abſurd. FIG.

COR. 2. Hence therefore if a right line meeting a conic ſection or the oppoſite hyperbola in two points be harmonically divided, or if in a diameter of the parabola two points equidiſtant from the vertex be aſſumed, and through the inner harmonic or equidiſtant point an ordinate be applied to the diameter drawn through the exterior harmonic or equidiſtant point, every right line drawn through this exterior point to meet the ſection or ſections in two points ſhall be harmonically divided in the point from which it is drawn, and in its concourſe with the ordinate.

For by the preceding COR. the right lines joining the exterior point and the vertices of the ordinate ſhall touch the ſection, and therefore the COR. is a caſe of the PROP.

COR. 3. If a right line meeting a conic ſection or the oppoſite hyperbolas in two points be harmonically divided, and in the exterior harmonic point meet a right line touching the ſection, the inner harmonic point and the point of contact ſhall be in a right line parallel to the ordinates applied to the diameter drawn through the concourſe.

COR. 4. If two right lines each meeting a conic ſection in two points, and meeting each other be harmonically divided, and the concourſe be one of the harmonic points common to each, the other two harmonic points ſhall be in a right line parallel to the ordinates applied to the diameter drawn through the concourſe.

Theſe two Corollaries are inferred in the ſame manner as the firſt.

COR. 5. If DE cutting a conic ſection or the oppoſite hyperbola in D, E, meet in Q a right line AQ, which touches the ſection in A, and R being the internal harmonic point in DE, an- 111.

FIG. ſwering to the external point Q; I ſay, that AR ſhall meet the ſection or ſections, and the right line drawn from Q to the point in which it meets it, ſhall touch the ſection.

Draw QB touching the ſection or oppoſite ſection in B (10.), and join AB. If AB do not paſs through R, but meet DE in ſome other point, in that other point and the point Q would DE be harmonically divided, though it be already harmonically divided in Q, R, which is abſurd.

COR. 6. If two right lines, the one touching, the other cutting an hyperbola or the oppoſite ſections in two points, meet in an aſſymptote, the harmonic point in the cutting line, which anſwers to the point of concourſe in the aſſymptote, ſhall be in a right line given in poſition, viz. in a parallel to the aſſymptote drawn through the point of contact.

COR. 7. Hence, if two or more right lines, cutting an hyperbola or the oppoſite ſections in two points, meet in an aſſymptote, the harmonic point in each of them, which anſwers to the concourſe with the aſſymptote, ſhall be in a right line given in poſition; viz. if from the concourſe a right line be drawn to touch the ſection, the harmonic point ſhall be in a right line drawn through the point of contact parallel to the aſſymptote.

COR. 8. Hence alſo, if a right line cutting an hyperbola or the oppoſite hyperbolas in two points, meet an aſſymptote, the right line parallel to the aſſymptote, and drawn through the harmonic point, which anſwers to the point of concourſe with the aſſymptote, ſhall meet the ſection, and the right line joining this point in the ſection and the concourſe with the aſſymptote, ſhall touch the ſection.

PROP. 38.

If a right line cutting a conic ſection or the oppoſite hyperbolas in two points be harmonically divided, the

the right lines touching the section in the points in which the harmonic line cuts it, and the right line drawn through either of the harmonic points parallel to the ordinates applied to the diameter drawn through the other harmonic point, shall verge to one common concourse.—But if the exterior harmonic point fall in an assymptote of the hyperbola, the right lines touching the hyperbola and a parallel to the assymptote drawn through the other harmonic point, shall verge to one common concourse. FIG.

CASE 1. When the right line cutting the section is a diameter of the ellipse or hyperbola.

Let the diameter DE of an ellipse or hyperbola meet each section in the vertices D, E, and be harmonically divided in O, S. The parallels to the ordinates applied to the diameter DE are parallel to DG, EG, touching the section in D, E (23. DEF.), viz. they are each parallel to each other, and therefore verge to one common concourse. 78.

CASE 2. When the right line is not a diameter of the conic section, and when the exterior harmonic point does not fall in an assymptote.

Let DE, not being a diameter, cut a conic section or the opposite hyperbolas, in D, E, and be harmonically divided in Q, R; I say, that the right line drawn through either of the points Q, R, parallel to the ordinates applied to the diameter drawn through the other point, shall pass through the concourse of the right lines touching the section in D, E. 124.

Because D, E, are not vertices of a diameter, the tangents at D, E, are not parallel (12.), and therefore do meet each other. Let them meet in G, and first, let Q be the interior harmonic point.—

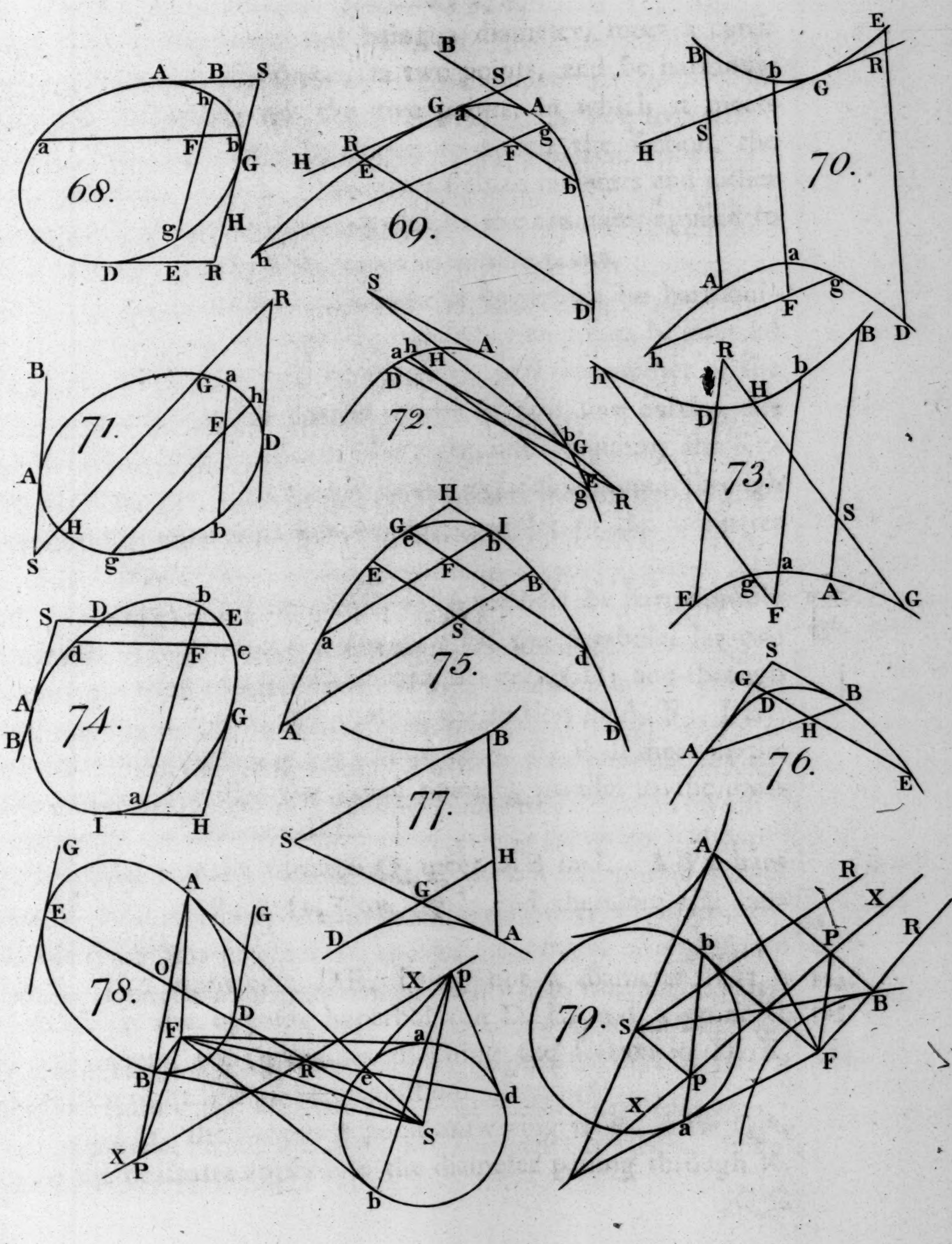
FIG. The right line drawn through Q, a point within the section, being parallel to the ordinates applied to a diameter, will meet the sections or section in two points (20 & 21. COR. DEF.). Let it meet the section or sections in A, B, and join AR, BR. Round any point O as a centre describe the generating circle, and draw every thing therein correspondent to what is drawn in the section. Because AB is drawn through the inner harmonic point Q parallel to the ordinates applied to the diameter drawn through R; AR, BR, will touch the section (1. COR. 37.), and therefore AR, BR, the focal correspondents to AR, BR, will touch the circle in A, B (9.). For the same reason will GD, GE, the focal correspondents to GD, GE, touch the circle in D, E. Wherefore in the circle, because DE is harmonically divided in Q, R, and AR, BR, touching the circle in A, B, meet in R, the right lines GD, GE, which touch the circle in D, E, will meet in AB (2. COR. LEM. 11.). Therefore their correspondents GD, GE, will also meet in AB, the conic correspondent to AB (1. COR. 4.), viz. AB passes through G.

125. Again, if Q be the exterior harmonic point, and QL be drawn parallel to the ordinates applied to the diameter drawn through R; I say, that GD, GE, shall meet in QL.

The same things remaining, because by the preceding, AB passes through G, therefore AB is harmonically divided in G, R (37.). But DE is also harmonically divided in Q, R, therefore QL passes through G (4. COR. 37.).

126. CASE 3. If the exterior harmonic point, viz. R, fall in an assymptote CK of the hyperbola; I say, that the tangents GD, GE, shall meet in QB, the right line drawn through Q parallel to the assymptote CK.

Every thing being drawn as in the preceding case, AB the correspondent to AB or QB will meet the generating circle in B, its concourse with the directrix (8. COR. DEF.), and BR the correspondent to CK, will pass through the same point B, and touch the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the circle therein (18. COR. DEF.). The remaining demonſtration FIG. is the ſame as in the preceding caſe, obſerving that AR, being joined, touches the ſection in A (6. COR. 37.).

COR. 1. If a right line, not being a diameter, meet a conic ſection or the oppoſite hyperbolas, in two points, and be harmonically divided, and if through the two points, in which it meets the ſection, two right lines be drawn to touch the ſection, the right line drawn through the concourſe of theſe tangents and either of the harmonic points ſhall be parallel to the ordinates applied to the diameter drawn through the other harmonic point.

COR. 2. If a diameter of an ellipſe or hyperbola be harmonically divided, or if in a diameter of a parabola two points be aſſumed equidiſtant from the vertex of the diameter; and from either of the harmonic or equidiſtant points be drawn a right line cutting the ſection or ſections in two points, the right lines touching the ſection in theſe two points ſhall meet in the right line drawn through the other harmonic or equidiſtant point, parallel to the ordinates applied to the diameter.

Let DE a diameter of an ellipſe or hyperbola be harmonically 135.
divided in Q, R; or being a diameter of the parabola, let two 136.
points Q, R, therein be equidiſtant from the vertex D; and through either point R be drawn RI cutting the ſection in A, B; I ſay, that the right lines touching the ſection in A, B, ſhall meet in the right line drawn through the other point Q parallel to the ordinates applied to the diameter DE.

Let the parallel drawn through Q meet AB in I. AB is harmonically divided in R, I (2. COR. 37.), and therefore the tangents at A, B, meet in QI.

COR. 3. If a right line DE, being not a diameter, cut a 124.
conic ſection or the oppoſite hyperbolas in D, E, and a point R 125.
be aſſumed therein, the right lines touching the ſection in D, E, ſhall meet in a right line given in poſition.

In DE find Q, the harmonic point anſwering to R, draw QG parallel to the ordinates applied to the diameter paſſing through R.

QG

FIG. QG is given in position, and by this Prop. the tangents at D, E, meet in QG.

Cor. 4. If a right line cutting an hyperbola in two points be harmonically divided, and the exterior harmonic point be in an aſſymptote, the concourſe of the right lines touching the ſection in the two points in which the harmonic line meets the ſection, and the inner harmonic point, ſhall be in a parallel to the aſſymptote.

Cor. 5. If two right lines touching an hyperbola meet each other, and through the concourſe be drawn a parallel to either aſſymptote, the right line joining the points of contact ſhall be harmonically divided in its concourſe with the parallel, and with the aſſymptote.

126. Let GE, GD, touching an hyperbola in E, D, meet in G, and through G be drawn GAB parallel to the aſſymptote CK, and ED be joined, meeting GA, CK, in Q, R; I ſay, that ED ſhall be harmonically divided in Q, R.

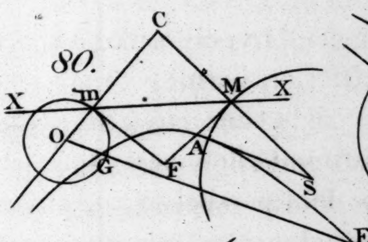
For, if not, let P be the harmonic point anſwering to R, then PG, being joined, ſhall be parallel to CK, viz. to QG, which is abſurd.

126. Cor. 6. The ſame things remaining, GR, being joined, ſhall be parallel to the ordinates applied to the diameter drawn through Q.

Becauſe ED is harmonically divided in R, Q, the parallel to theſe ordinates drawn through R, does by this Prop. paſs through G.

Cor. 7. If two right lines each meeting a conic ſection or the oppoſite hyperbolas in two points meet each other, and if the right lines touching the ſection in the points in which one of the right lines meets it, have their concourſe in the other, the right lines touching the ſection or ſections in the points in which the other right line meets the ſection or ſections, ſhall alſo have their concourſe in the alternate line.

124. Let AB, DE, cutting a conic ſection or the oppoſite hyperbolas in A, B, and D, E, meet each other in Q, and the right lines touching the ſection or ſections in A, B, have their concourſe R in DE;



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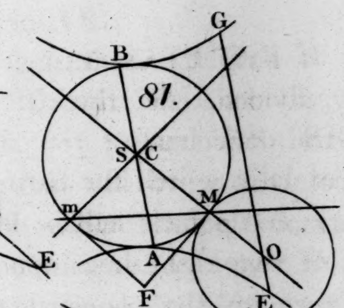

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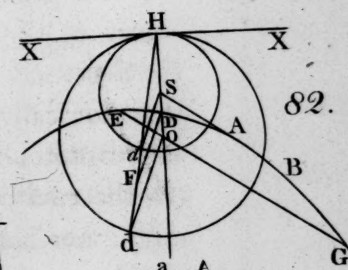

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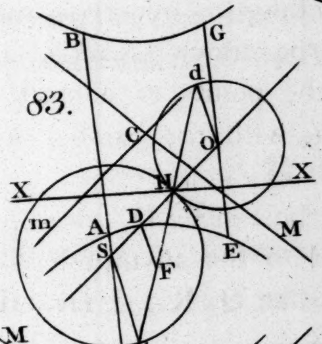

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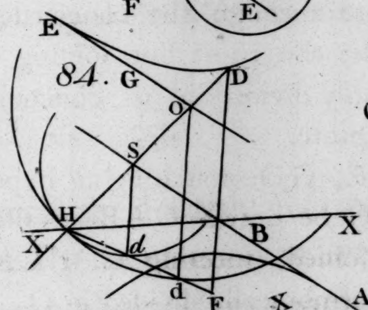

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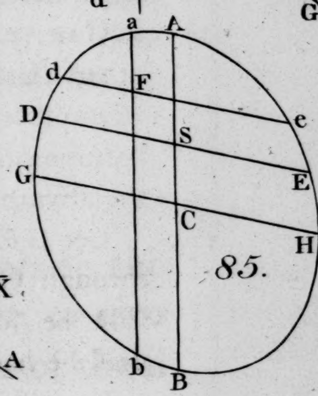

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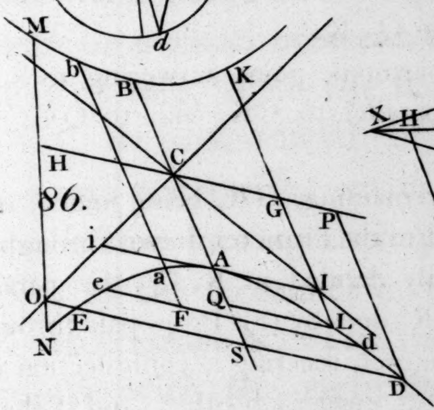

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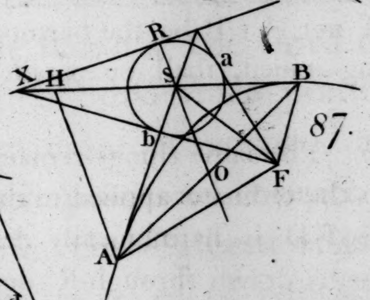

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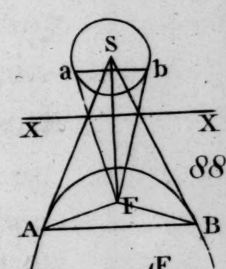

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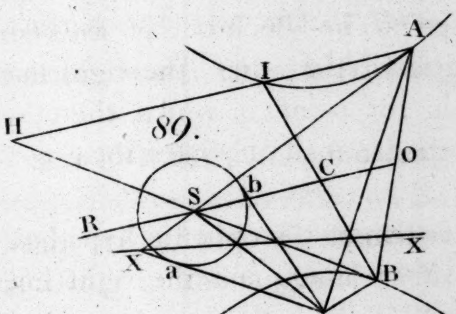

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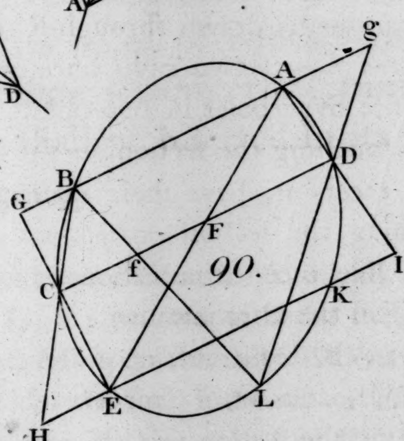

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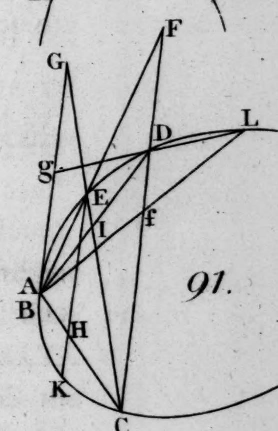


91.

DE; I say, the right lines touching the section or sections in D, E, FIG.
shall also have their concourse in AB.

In AB find the point G, such that AB shall be harmonically divided in Q, G (LEM. 3.). Because DE meeting AB in Q is harmonically divided in Q, R (35.), and AR, BR, touch the section in A, B, the right line AB will be parallel to the ordinates applied to the diameter which passes through R (3. COR. 37.), and therefore the right lines touching the section or sections in D, E, will have their concourse in AB, by the 1. COR. of this Prop.

PROP. 39.

If three right lines touching a conic section or the opposite hyperbolas mutually meet each other, or two of them be parallel between themselves, each of them in the first case, or in the second, the third touching line, shall be harmonically divided in the point of contact, in its concourses with the other two, and with the right line joining their points of contact.

Also, in the second case, the segment of either parallel intercepted between its point of contact and its concourse with the right line joining the points of contact of the other two, shall be bisected in its concourse with the third tangent.

Let EG, EF, FG, touching a conic section in A, B, D, either 127.
mutually meet each other in E, G, F; or two of them EF, FG, 128.
being parallel between themselves, fall upon the third EG in E, G. If either of the tangents, as EG in the first instance, or EG the third tangent in the second instance, meet the right line BD in H; I say, that FG shall be harmonically divided in A, H.

From

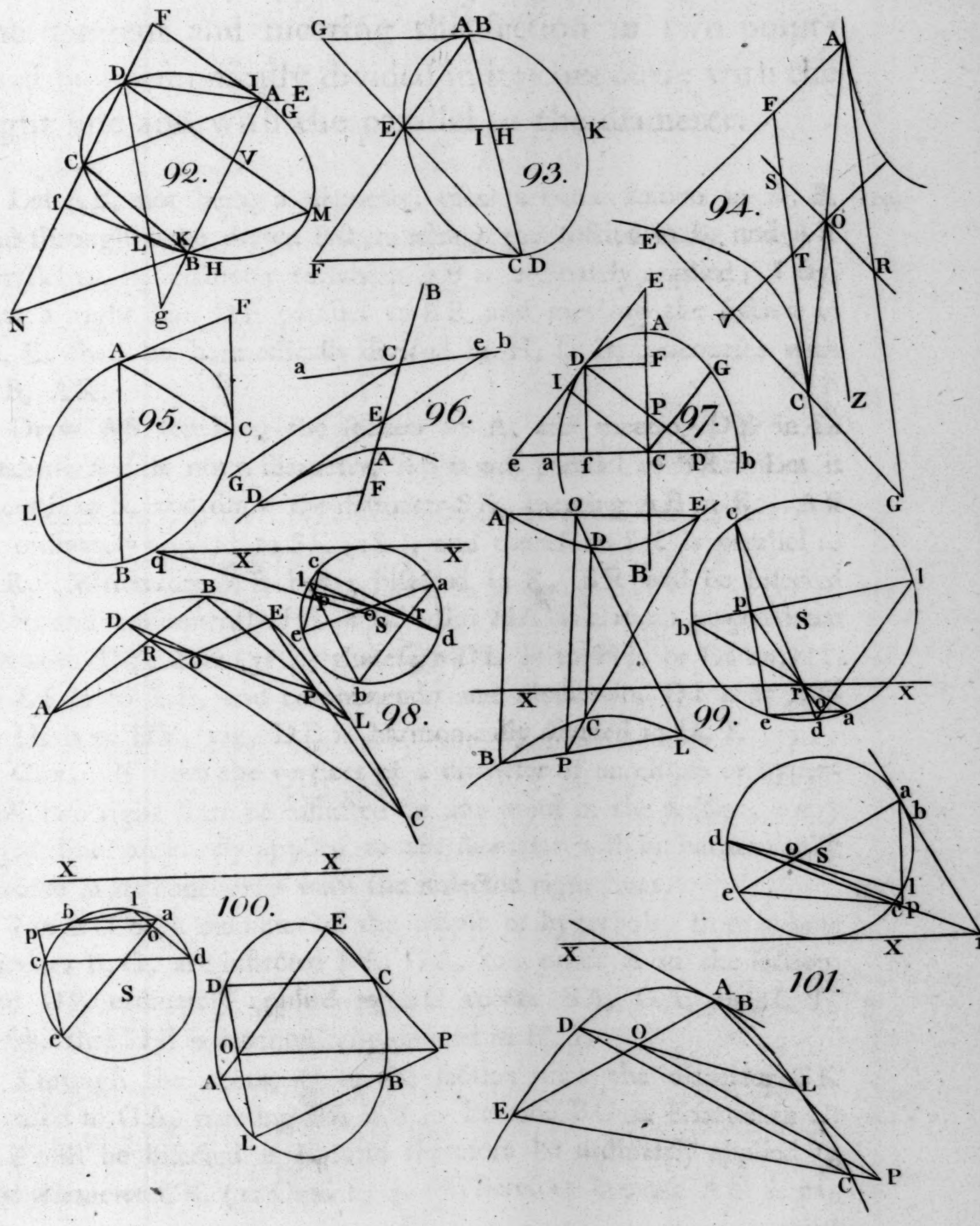
FIG. From A draw AF to the concourse of EF, GF; or when EF, GF, are parallel, draw AF parallel to either of them, and let AF meet BD in K, and the section again in I. Join HI, and draw HF to the concourse F, or parallel to EF, GF.—Because EF, GF, touching the section in B, D, meet in F, the right line FK which meets the section in A, I, and BD in K, will be harmonically divided in A, I (37); therefore the tangents at A, I, meet in BD (38.). But EG touching the section in A does meet BD in H, therefore HI also touches the section in I. And when EF, GF, AF, are mutually parallel, because BD is then a diameter of the section (12.), AF will be ordinately applied to BD (23. DEF.), and therefore as before, HI will touch the section in I. Wherefore because HA, HI, touch the section, HK will be harmonically divided in B, D (35.), and consequently FB, FK, FD, FH, are harmonicals. But EG falls upon these harmonicals in E, A, G, H; therefore EG is also harmonically divided in A, H (1. COR. LEM. 4.).

128. CASE 2. When two of the tangents FE, FG, are parallel, if either of them FG meet the right line AB joining the other two points of contact in L, the portion DL shall be bisected in G, its concourse with EG.

Other things remaining the same, join AD. Because HK is harmonically divided in B, D, AB, AK, AD, AH, are harmonicals, and therefore FG, parallel to one of them AK, and falling upon the other three, will be bisected in the middle concourse (LEM. 4.), viz. DL is bisected in G.

PROP. 40.

If a right line, not being a diameter, meet a conic section in two points, and through one of the points be drawn a right line touching the section, through the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the other a parallel to the diameter to which the right line is ordinately applied, every right line parallel to the tangent and meeting the ſection in two points ſhall be harmonically divided in its concourſe with the right line and with the parallel to the diameter. FIG.

Let AB, not being a diameter, meet a conic ſection in A, B, and through B be drawn BR touching the ſection in B, and AR parallel to the diameter to which AB is ordinately applied; I ſay, that a right line DE parallel to BR and meeting the ſection in D, E, ſhall be harmonically divided in H, I, its concourſes with AB, AR. 129.

Draw AS touching the ſection in A, and meeting DE in L. Becauſe AB is not a diameter, AS is not parallel to BR. Let it meet it in S, and draw the diameter SK, meeting AB in K. AB is ordinately applied to SK (18.), and therefore SK is parallel to AR. Wherefore AB being biſected in K, BR will be biſected in S, and conſequently HI in L. But HL is a mean proportional between DL, LE (31.); therefore DL is to HL or LI as HL or LI is to LE, and componendo and dividendo, DI is to DH as IE is to HE, viz. DE is harmonically divided in H, I.

COR. If from the vertices of a diameter of an ellipſe or hyperbola two right lines be inflected to any point in the ſection, every right line ordinately applied to the diameter will be harmonically divided in its concourſes with the inflected right lines.

Let BG be a diameter of the ellipſe or hyperbola, from whoſe vertices B, G, are inflected BA, GA, to a point A in the ſection, and DE ordinately applied to BG meets BA, GA, in H, I; I ſay, that DE is harmonically divided in H, I.

Through the centre C of the ſection draw the diameter CK parallel to GA, meeting BA in K. Becauſe BG is biſected in C, AB will be biſected in K, and therefore be ordinately applied to the diameter CK (1. COR. 15.). Wherefore becauſe AG is pa-

 rallel

FIG. rallel to the diameter to which AB is ordinately applied, and DE being ordinately applied to the diameter BG, is parallel to the tangent at the vertex B (23. Def.), it will be harmonically divided in H, I.

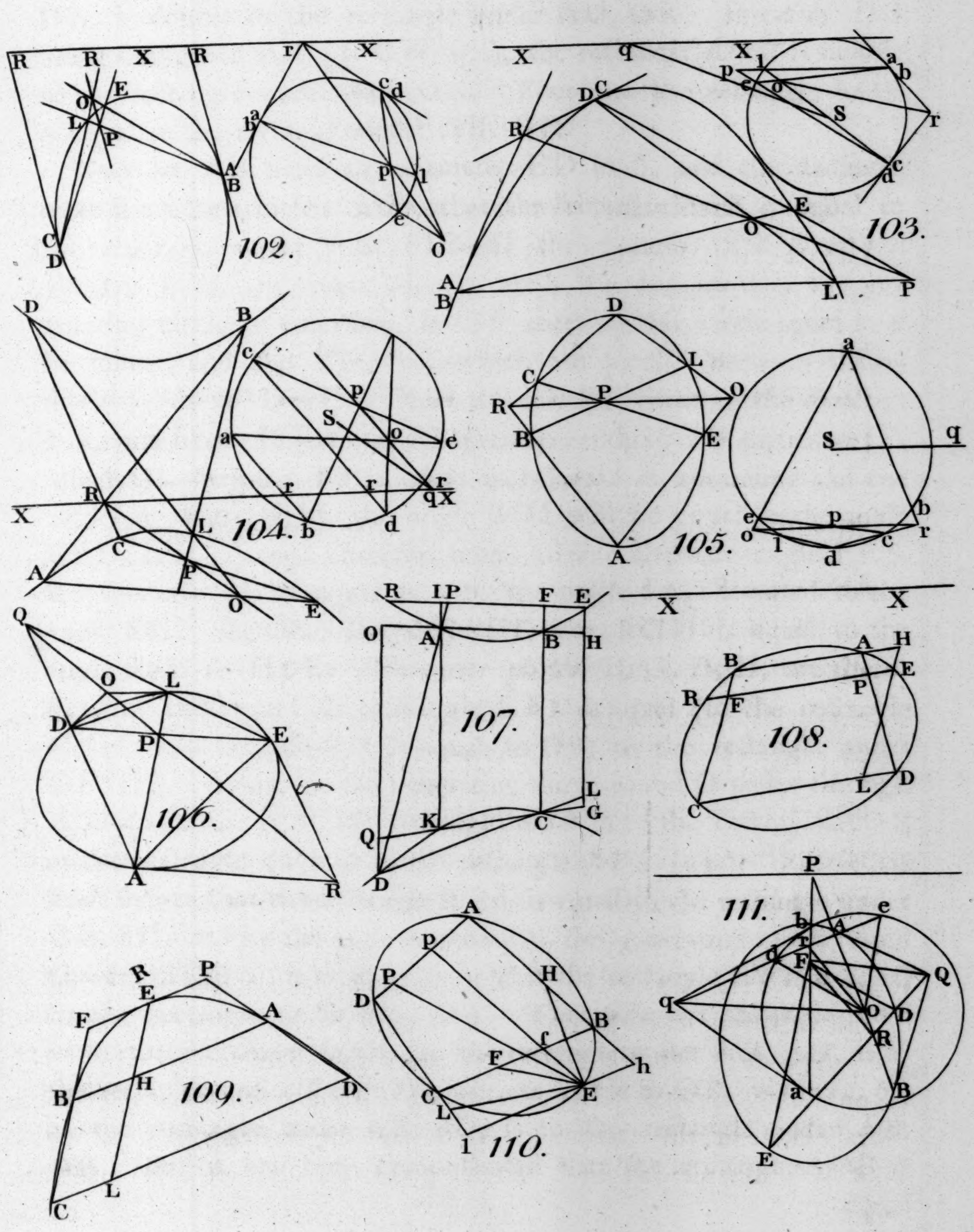
PROP. 41.

In a parabola, if a right line touching or cutting the ſection meet a diameter, the ſquare of the touching, or the rectangle under the ſegments of the cutting line, intercepted between the ſection and the diameter, ſhall be equal to the rectangle under the parameter of that diameter to whoſe ordinates the touching or cutting line is parallel, and the abſciſs of the diameter which it meets, intercepted between its vertex and the right line.

In the ellipſe and hyperbola, if a right line be ordinately applied to any diameter of the former, or to a tranſverſe diameter of the latter; and a fourth proportional be taken to the diameter, to either abſciſs, and to the parameter of the diameter; the ſquare of the ſemi-ordinate ſhall be equal to the rectangle under the other abſciſs and the fourth proportional.

Part I. Case I. In the parabola, when the right line is drawn through the focus.

142. And firſt, let the right line meet the diameter to which it is ordinately applied. Let AB drawn through the focus F of a parabola meet the ſection in A, B, and the diameter PR to which it is ordinately applied, in the point O, alſo let PR meet the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section in R. AB is the parameter of PR (24. DEF.), and I say, that the rectangle AOB is equal to the rectangle under AB, OR.

Because AB is double to AO (15.), the rectangle under AB, OR, is double to the rectangle under AO, OR. But also, OB being double to OR (5. COR. 17.), the rectangle AOB is double to the rectangle under AO, OR. Therefore the rectangle AOB is equal to the rectangle under AB, OR.

Now let AB meet any diameter ED in S, and the diameter meet the section in D; I say, that the rectangle ASB is equal to the rectangle under AB, SD.—If the diameter SD meet the directrix in E, and round S, viz. with the distance SE, the generating circle be described, let EF meeting this circle again in D be joined, and also FD, SD, which are parallel between themselves (3. & 18. DEF.). Draw the axis FI, meeting the directrix in I, and SD in H, and in FI produced taking IG equal to IF, join EG.—Because FI is equal to IG, EI is common, and the angles at I are right, the angle EGI will be equal to the angle EFI (4. e. 1.), and therefore equal to the alternate angle FES. But because SE is equal to SD, the angle SDE is equal to the angle FES, therefore the angle EGI or EGH is equal to the angle SDE or HDE. The four points E, G, D, H, are therefore in a circle, and the rectangle EFD is equal (to the rectangle GFH, viz. because FH is equal to DS) to the rectangle under GF, DS. Again, as the generating circle round O passes through A, B (1. COR. 17.), the rectangle ASB is to the rectangle AOB as the rectangle EFD is to the rectangle AFB (22.). But it has been shewn that the rectangle EFD is equal to the rectangle under GF, SD, and by the same reasoning if the generating circle round O were described, it would appear that the rectangle AFB is equal to the rectangle under FG, OR. Therefore the rectangle ASB is to the rectangle AOB (as the rectangle under FG, SD, is to the rectangle under FG, OR, viz. as SD is to OR, viz. 1. e. 6.) as the rectangle under AB, SD, is to the rectangle under AB, OR. But it has been demonstrated that the rectangle AOB is

 equal

FIG. equal to the rectangle under AB, OR, therefore the rectangle ASB is equal to the rectangle under AB, SD.

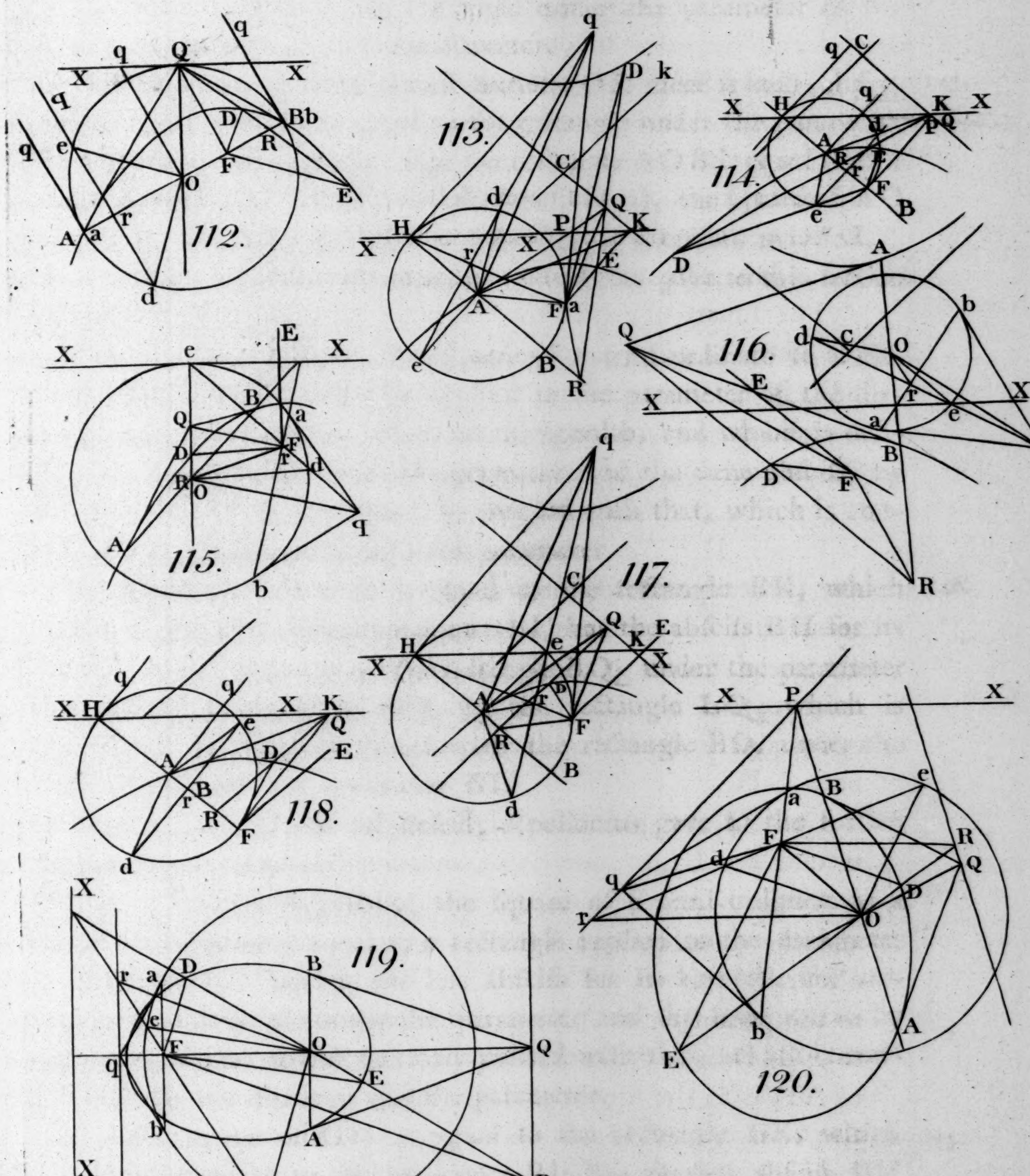
Case 2. Let the right line not pass through the focus.

143. Let AB, either touching a parabola in A, or cutting it in A, B,
144. and not passing through the focus, meet a diameter ED in S. Draw AFB through the focus F, parallel to AB, meeting the section in A, B, and ED in s. Then AB is the parameter of the diameter to whose ordinates AB is parallel (24. Def.), therefore D being the vertex of the diameter ED; I say, that the square of AS or the rectangle ASB is equal to the rectangle under AB, SD.

The square of AS or the rectangle ASB is to the rectangle ASB (as SD is to sD (5. Cor. 22.), viz.) as the rectangle under AB, SD, is to the rectangle under AB, sD. But the rectangle ASB is equal to the rectangle under AB, sD, therefore the square of AS or the rectangle ASB is equal to the rectangle under AB, SD.

145. Part II. Let AGB be an ellipse or hyperbola, ACB, DCE,
146. any two conjugate diameters, of which in the hyperbola AB is the transverse, GH an ordinate applied to AB, meeting it in H, and BL be a fourth proportional to AB, either abscis AH, and the latus rectum of AB. I say, that the rectangle under BL and the other abscis BH shall be equal to the square of the semi-ordinate GH.

Let BL be placed at right angles to AB, and compleat the rectangle HBLK; join AK, meeting BL in I, also compleat the rectangle BO. Because BL is a fourth proportional to AB, AH, and the parameter of AB, and on account of the parallels, BL is a fourth proportional to AB, AH, and BI, therefore BI is equal to the parameter of AB; and consequently is a third proportional to AB, DE (25. Def.). Wherefore AB is to BI (as the square of AB is to the square of DE, viz.) as the rectangle AHB is to the square of GH (6. Cor. 23.) But also AB is to BI (as AH is to BL, viz.) as the rectangle AHB is to the rectangle HBL (1. e. 6.). Wherefore, ex æquo, the rectangle AHB is to the square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ſquare of GH as the rectangle AHB is to the rectangle HBL, and FIG.
conſequently the ſquare of GH is equal to the rectangle HBL.

COR. 1. In a parabola, the ſquare of a ſemi-ordinate applied to any diameter is equal to the rectangle under the parameter of the diameter and the abſciſs of the diameter.

Let AB ordinately applied to a diameter OR meet it in O; I ſay, 143.
that the ſquare of AO is equal to the rectangle under the parameter of OR and the abſciſs OR. For the rectangle AOB is equal to this rectangle, and becauſe AB is biſected in O (15.), the ſquare of AO is equal to the rectangle AOB. The Cor. is therefore manifeſt.

For this reaſon, viz. of equality, Apollonius gave to this ſection the name of PARABOLA.

COR. 2. In an ellipſe, the ſquare of a ſemi-ordinate to a diameter is equal to a rectangle applied to the parameter of the diameter, and having either abſciſs for its breadth, and which is deficient of the rectangle under the parameter and the ſame abſciſs, by a rectangle ſimilar to and ſimilarly ſituated with that, which is contained under the diameter and the parameter.

For the ſquare of GH is equal to the rectangle BK, which 146.
rectangle is applied to the parameter BI, has the abſciſs BH for its breadth, and is deficient of the rectangle BQ under the parameter BI and the ſame abſciſs BH, by the rectangle LQ, which is ſimilar to and ſimilarly ſituate with the rectangle BO, under the diameter AB and the parameter BI.

For this reaſon, viz. of defect, Apollonius gave to the ſection the name of ELLIPSE.

COR. 3. In an hyperbola, the ſquare of a ſemi-ordinate to a tranſverſe diameter is equal to a rectangle applied to the parameter of the diameter, having the leſs abſciſs for its breadth, and exceeding the rectangle under the parameter and the ſame abſciſs by a rectangle ſimilar to and ſimilarly poſited with that, which is contained under the diameter and the parameter.

For the ſquare of GH is equal to the rectangle BK, which 145.
rectangle is applied to the parameter BI, has the leſs abſciſs BH for its breadth, and exceeds the rectangle BQ under the parameter

FIG. BI and the same abscis BH, by the rectangle LQ, which is similar to and similarly posited with the rectangle BO, under the diameter AB and the parameter BI.

For this reason, viz. of excess, Apollonius gave to this section the name of HYPERBOLA.

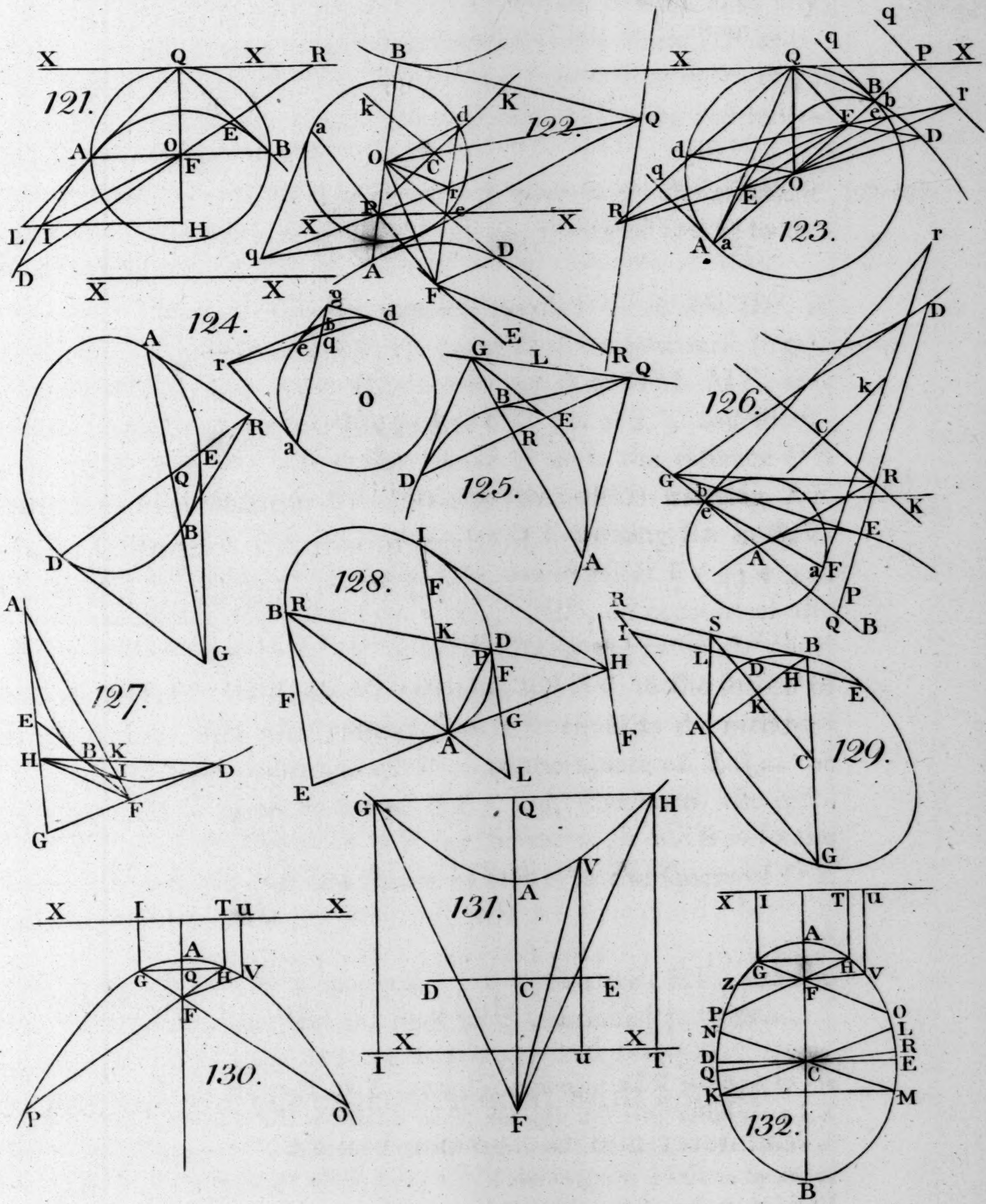
NOTE. This character of equality, defect or excess, discriminates the three sections in their principal common properties; which is observable in the fundamental property, expressed in the 2. DEF., and also in the property of the generating circle, touching, falling short of, or cutting, the directrix.

PROP. 42.

If from the vertex of any diameter of a conic section a right line be drawn ordinately applied to an axis of the section, and from the point in which the ordinate meets the section again a right line be drawn perpendicular to the ordinates applied to the diameter; then in the ellipse and hyperbola the rectangle under the segments of the diameter intercepted between its vertices and the perpendicular shall be to the square of the perpendicular, intercepted between the diameter and the point in the section, as the square of the diameter is to the square of its conjugate.

But in the parabola, the segment of the diameter intercepted between the vertex and the perpendicular, shall be a third proportional to the latus rectum of the diameter and the perpendicular.

196, 197. Let YBU be a conic section, whose axes are Yy, Uu, or if
198. the section be a parabola, whose single axis is OU. In the ellipse and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and hyperbola, let O be the centre of either section, and BOb, XOx, any two conjugate diameters, and in the parabola Bb, any diameter. From B the vertex of the diameter Bb draw BF ordinately applied to the axis Uu, and meeting the section again in F. Draw FE perpendicular to the ordinates applied to Uu, and meeting Bb in E. I say, FIG.

1. That in the case of the ellipse and hyperbola, the rectangle bEB shall be to the square of EF as the square of OB is to the square of OX. 196, 197.

Join OF. Because BF is ordinately applied to an axis Uu, it is bisected by it at right angles in the point M of its concourse (15.). Wherefore OM, MB, being respectively equal to OM, MF, and the angles at M equal, OB is equal to OF (4. e. 1.), and therefore a circle described round the centre O with the distance OB or Ob, will pass through F. Describe this circle meeting EF again in f, and draw FZ, $f$$z$, parallel to OX meeting Bb in Z, z. Because OX, being perpendicular to EF, would bisect Ff (3.e.3.), Zz will be bisected also in O (10. e. 6.). Also, on account of the parallels, fE is to EF as zE is to EZ, and the rectangle fEF is to the square of EF as the rectangle zEZ is to the square of EZ (1. e. 6.). But the rectangle bEB is equal to the rectangle fEF, therefore the rectangle bEB is to the square of EF as the rectangle zEZ is to the square of EZ; and, dividendo, the rectangle bEB is to the square of EF (as the rectangle bZB is to the square of FZ*, viz.) as the square of OB is to the square of OX (6. COR. 23.).

II. If the section be a parabola; I say, that BE, EF, and the latus rectum of the diameter Bb shall be in continued proportion.

* The excess of the rectangle zEZ above the rectangle bEB is equal to the rectangle bZB.—Because bB is bisected in O, and also zZ, the rectangle zEZ together with the square of OZ is equal (to the square of EO, viz.) to the square of OB together with the rectangle bEB (6. e. 2.) Take away the common square of OZ, and the rectangle zEZ is equal to the rectangles bZB, bEP, together (5. e. 2.)

Let

FIG. Let L be the latus rectum of the diameter bB, and draw FZ parallel to the ordinates applied to bB. Then because BF is ordinately applied to the axis, it will be perpendicular to EZ. Therefore FZ is a mean proportional between EZ, BZ, and it is also a mean proportional between L, BZ (1. COR. 41.); wherefore EZ is equal to L. But EF is a mean proportional between EZ, EB; therefore EF is a mean proportional between L, EB.

COR. 1. If a diameter Bb of an ellipse or hyperbola be harmonically divided in E, D; E being the external point, Xx, the diameter conjugate to Bb, O the centre of either section; and if DG ordinately applied to Bb, meet the section in G, while from E is drawn EF perpendicular to the ordinates applied to Bb, and therein the point F taken, whether it fall in the section or no, such that the rectangle bEB be to the square of EF as the square of OB is to the square of OX; I say, that OD shall be to OB or OB to OE as DG is to EF.

For, the square of DG is to the rectangle bDB (as the square of OX is to the square of OB (6. COR. 23.) viz.) as the square of EF is to the rectangle bEB. But as bB is harmonically divided in D, E, the rectangle bDB shall be equal to the rectangle ODE, and the rectangle bEB to the rectangle OED (LEM. 5.). Wherefore the square of DG is to the square of EF (as the rectangle ODE is to the rectangle OED, viz.) as OD is to OE. Again, because OB is a mean proportional between OD, OE; OD is to OE as the square of OD is to the square of OB. Therefore, ex æquo, the square of OD is to the square of OB as the square of DG is to the square of EF, and OD is to OB or OB is to OE as DG is to EF.

COR. 2. The same things remaining as in the Prop., if in the diameter bB, the point D be assumed such that BD is to bD as EB is to Eb, when the section is an ellipse or hyperbola, or in the case of the parabola, if BD be made equal to BE; and in each be drawn DG ordinately applied to bB, and meeting the section in G; I say, that FD being joined, shall be equal to DG.

In the ellipse and hyperbola, because bB is harmonically divided in D, E, and DF, EF are drawn to a point F in the semicircle described on bB, DF will be to EF (as DB is to BE (3. Cor. Lem. 6.) viz. as OD is to OB (1. Cor. Lem. 5.) viz.) as DG is to EF (1. Cor.). Wherefore DG is equal to DF. FIG.

In the parabola, because EF is a mean proportional between L, EB, and DG is a mean proportional between L, BD (1. Cor. 41.), that is, between L, EB; therefore EF is equal to DG. But because EB, BF, are severally equal to DB, BF, and the angles at B are right, EF will be equal to DF (4. e. 1.). Therefore DF is equal to DG.

PROP. 43.

A conic section cannot meet another in more than four points; if a conic section touch another, it can meet it only in two points more; and if two conic sections touch each other in two points, they cannot meet in any other point.

Part 1. Let two conic sections meet each other in the four points A, B, C, D; I say, that they cannot meet each other in any other point.

If possible, let them meet each other in a fifth point E. Join AB, BC, CD, AD, and draw EP parallel to AB meeting AD in P, also EL parallel to BC, meeting CD in L. Then EP is to EL in a given ratio (3. Cor. 26.). From A draw any right line to meet the two sections, towards the same parts of E, in any two points D, *d*, and meeting EP in P, also join CD, C*d*, meeting EL in L, *l*. Then for the same reason Ep is to El, and Ep is to E*l*, in the same given ratio, in which EP is to EL. Therefore El is equal to E*l*, which is absurd, because the points 147.

FIG. L, *l*, falling towards the ſame parts of E, the one is but a part of the other. Therefore, the conic ſections do not meet in a fifth point E.

148. PART 2. Let two conic ſections touching each other in the point B or C, meet each other in two other points A, D; I ſay, that they cannot meet again.

If poſſible, let them meet in a fourth point E. Draw the right line BC touching each of the ſections in the common point of contact (9. DEF.), join AB, CD, AD, and as before draw EP, EL, parallel to AB, BC, meeting AD, CD, in P, L. Draw alſo from A any right line meeting the two ſections, towards the ſame parts of A, in D, *d*, and meeting EP in P; and join CD, C*d*, meeting EL in L, *l*. Then as before, EP is to EL in a given ratio, and in the ſame given ratio is EP to EL, and EP to E*l*. Therefore EL is equal to E*l*, which is abſurd for the ſame reaſon as above, and conſequently the ſections do not meet in a fourth point.

149. PART 3. If two conic ſections touch each other in the point A or D, and alſo touch each other in another point B or C; I ſay, that they do not meet again.

If poſſible, let them meet in a third point E. Draw AD, BC, touching each ſection in the common points of contact A or D, and B or C. Join AB or CD, and draw EP, EL, parallel to AB, BC, meeting AD, CD, in P, L; alſo, as before, draw any right line from A to meet the two ſections in D, *d*, towards the ſame parts of A, and meeting EP in P, and join CD, C*d*, meeting EL in L, *l*. Then in like manner, EP is to EL in a given ratio, and in the ſame given ratio is EP to EL, and EP to E*l*. Therefore EL is equal to E*l*, which is abſurd, as above, and therefore the ſections do not meet each other in a third point.

PROP.

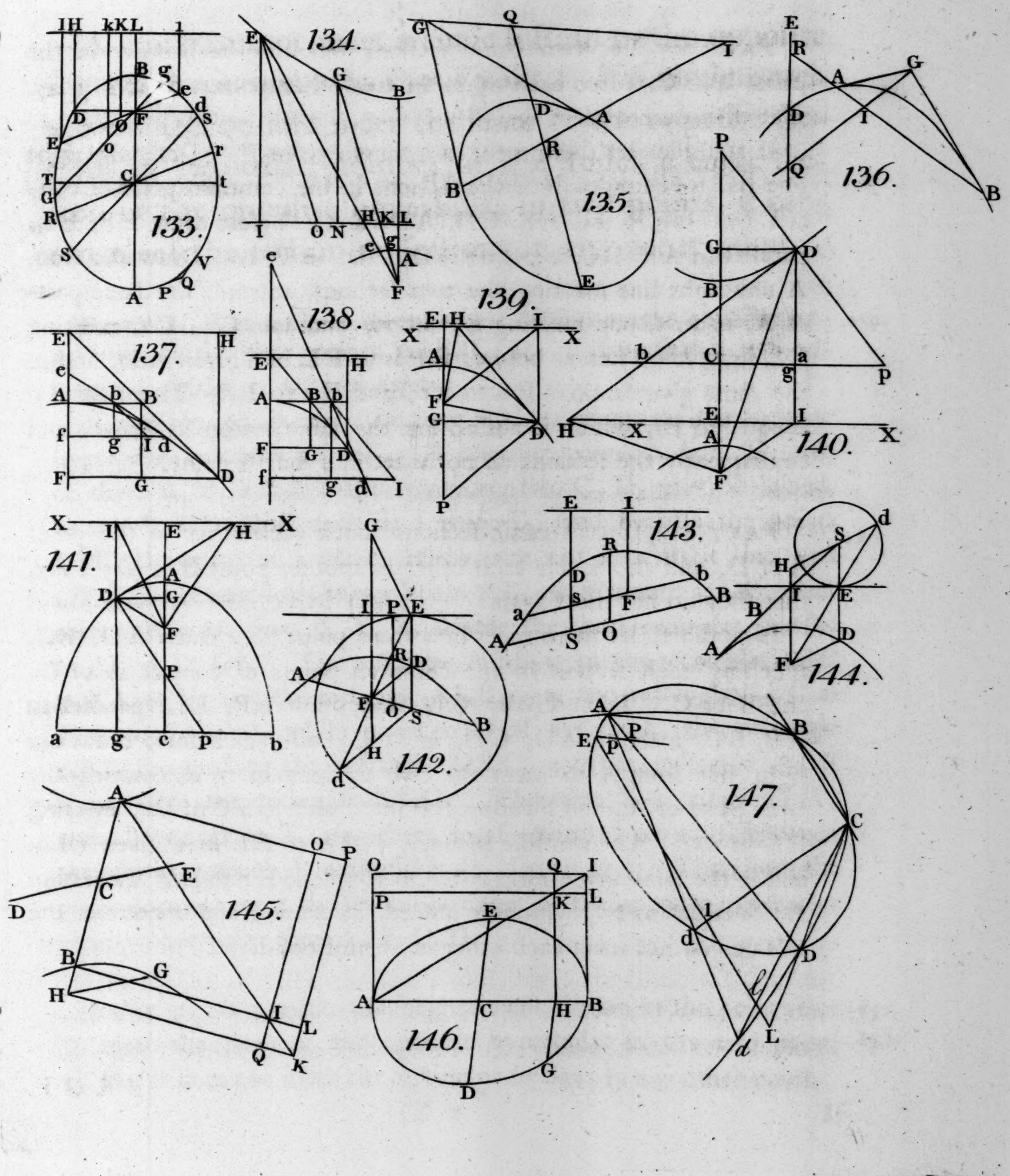
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FIG.

PROP. 44.

A circle cannot meet a conic section or the opposite hyperbolas in more than four points; if a circle touch a conic section and meet the same or the opposite sections in two other points; or if it touch a conic section or the opposite hyperbolas in two points, it cannot meet the section or sections in any other point.

PART 1. Let a circle meet a conic section or the opposite hyperbolas in the four points A, B, C, D; I say, that it will not meet the section or sections again. 150.

If possible, let it meet the same section, or one of the opposite hyperbolas in a fifth point K. Of the four points A, B, C, D, two of them A, B, and also the remaining two C, D, may be joined, so as AB, CD, shall meet in a point E; and because the points C, K, D, being in a conic section, are not in a strait line, and CK, KD, cannot both be parallel to AB, one of them, as CK, must meet AB in L. Let X, Y, Z, denote the focal ordinates parallel to AB, CD, CK. The rectangle AEB is to the rectangle DEC as X is to Y, and the rectangle ALB is to the rectangle KLC as X is to Z (23.), while on account of the circle, the rectangle AEB is equal to the rectangle DEC, and the rectangle ALB is equal to the rectangle KLC. Therefore X is equal to Y, and also equal to Z, viz. three focal ordinates are equal between themselves, which is impossible (1. COR. 27.). Therefore the circle does not meet the section or either of the opposite hyperbolas in any other point than A, B, C, D.

PART 2. If a circle touching a conic section in the point A or B, meet the same or the opposite hyperbolas in the two points C, D; it does not meet the section or sections in any other point. 152. 153.

FIG. If possible, let it meet the same section or either hyperbola again
153. in K; draw AB touching the circle and section in the common point of contact A or B, and join CD. If AB be parallel to CD, join CK, DK, meeting AB in L, H, and let X, Z, U, represent the focal ordinates parallel to AB, CK, DK. The square of AL is to the rectangle CLK as X is to Z, and the square of AH is to the rectangle DHK as X is to U (23.). But on account of the circle the square of AL is equal to the rectangle CLK, and the square of AH is equal to the rectangle DHK; therefore X is equal to Z, and also to U, viz. as before, X, Z, U, are equal between themselves, which is impossible, and therefore the circle does not meet the section or sections in any other point.

152. And if AB meet CD in the point E, because CK, DK, cannot both be parallel to AB, one of them, as CK, will meet AB in L. Then as in Part 1. the square of AE is to the rectangle CED as X is to Y, and the square of AL is to the rectangle CLK as X is to Z, wherefore X, Y, Z, as in Part 1. will be equal between themselves, which being impossible, the circle does not meet the section or an opposite hyperbola in any other point.

154. PART 3. If a circle touch a conic section or the opposite hy-
155. perbolas in two points A, C; it does not meet the section or either hyperbola in any other point.

If possible, let it meet the same section or an opposite hyperbola in K. Draw AB, CD, touching the circle and section in the
154. points of common contact A, C. If AB be parallel to CD, join AK, CK, meeting CD, AB, in H, L, and let X, Z, U, denote the focal ordinates parallel to AB or CD, and to CK, AK. The square of AL is to the rectangle CLK as X is to Z, and the square of CH is to the rectangle AHK as X is to U (23.). But on account of the circle, the square of AL is equal to the rectangle CLK, and the square of CH to the rectangle AHK, therefore, as above, X, Z, U, are equal to each other, which is im-
155. possible, and consequently the circle does not meet the section or an opposite hyperbola in any other point.

If

If AB meet CD, let it meet it in E. Either CK or AK will meet the alternate tangent; let CK meet AB in L, and X, Y, Z, denote the focal ordinates parallel to AB, CD, CK. The square of AE is to the square of CE as X is to Y, and the square of AL is to the rectangle CLK as X is to Z; while from the property of the circle, the square of AE is equal to the square of CE, and the square of AL is equal to the rectangle CLK; therefore X, Y, Z, are equal between themselves. But this is impossible, therefore the circle does not meet the section or an opposite hyperbola in any other point. FIG. 155.

PROP. 45.

If a circle cut a conic section or the opposite hyperbolas in four points, and two right lines be drawn joining the points of section, two by two; then if the two right lines be parallel, they shall each be parallel to the ordinates applied to an axis of the section; but if they meet each other, the right line bisecting the angle formed by their concourse shall be parallel to an axis of the section.

Again, if a circle touch a conic section, and meet the same or an opposite hyperbola in two other points, and the right line touching each in the common point of contact, as also the right line joining the two points of concourse be drawn; then if these two right lines be parallel, they shall each be parallel to the ordinates applied to an axis of the section; but if they meet each other, the right line bisecting the angle at their concourse shall be parallel to an axis of the section, or to the ordinates applied to an axis.

Lastly,

FIG. Lastly, if a circle touch a conic section or the opposite hyperbolas in two points, and the right lines touching each in the common points of contact be drawn; then if the touching lines be parallel, they shall each be parallel to an axis of the section, and the right line joining the points of contact shall be an axis of the section; but if they meet, the right line bisecting the angle at their concourse shall be an axis of the section.

151. Case 1. If a circle cut a conic section in four points, A, B, C, D; then AB, CD, being joined,

First, let AB, CD, be parallel between themselves; I say, that they are each parallel to the ordinates applied to an axis.

Bisect AB in P, and draw PO perpendicular thereto, meeting CD in O. PO passes through the centre of the circle (Cor. 1. e. 3.), and being also perpendicular to CD, it will bisect it in O (3. e. 3.) Wherefore PO bisecting the parallels AB, CD, inscribed in a conic section, or in the opposite hyperbolas, is a diameter of the section, to which AB, CD, are ordinately applied (3. Cor. 15.). But they meet PO at right angles, therefore PO is an axis of the section, to which AB, CD, are ordinately applied (2. Cor. 13.).

150. Secondly, let AB, CD, meet each other in E; I say, that EI bisecting the angle AED shall be parallel to an axis of the section, or to the ordinates applied to an axis.—Let X, Y, represent the focal ordinates parallel to AB, CD. Then the rectangle AEB is to the rectangle CED as X is to Y (23.), and on account of the circle, the rectangle AEB is equal to the rectangle CED; therefore X is equal to Y, and an axis of the section bisects the angle contained by X, Y (1. Cor. 27.). But because X, Y, are parallel to

to AB, CD, the right line which bisects the angle formed by X, Y, will be parallel to EI, which bisects the angle AED. FIG.

Therefore EI is parallel to an axis of the section, or to the ordinates applied to an axis.

CASE 2. Let a circle touch a conic section in A, and cut it in the two points C, D, and let AB touching the circle and section in A be drawn, and CD be joined. Then, 152, 153.

First, if AB be parallel to CD; I say, that each shall be parallel to an axis, or to the ordinates applied to an axis of the section. Draw AO perpendicular to AB and CD. Because AB touches the circle in A, AO will be a diameter of the circle (19. e. 3.), and therefore will bisect CD in O (3. e. 3.). Wherefore as AB, CD, are two right lines, of which one AB touches the section in A, the other CD meets it in the points C, D, the right line AO drawn from the point of contact to bisect CD, will be a diameter of the section (4. COR. 15.), and being also at right angles to CD, which is ordinately applied to it, it will be an axis of the section (2. COR. 13.). AB, CD, are therefore parallel to an axis, or to the ordinates applied to an axis of the section. 153.

Secondly, if AB meet CD in the point E; I say, that EI, bisecting the angle AED, shall be parallel to an axis of the section. 152.

Let X, Y, denote the focal ordinates parallel to AB, CD. The square of AE is to the rectangle CED as X is to Y (23.). But on account of the circle, the square of AE is equal to the rectangle CED, therefore X is equal to Y, and consequently for the same reason as in the second part of CASE 1. EI will be parallel to an axis of the section.

CASE 3. Let a circle touch a conic section or the opposite hyperbolas in two points A, C, and AB, CD, be drawn to touch the circle and section in the common points A, C. Then, 154, 155.

First, if AB, CD, be parallel to each other; I say, that AB, CD, shall each be parallel to an axis of the section. 154.

Because

FIG. Because AB, CD, parallel to each other, touch both the ſection and the circle, the points A, C, will be diametrically oppoſite in each (19. e. 3. & 12.); viz. AC, being joined, is a diameter of each, and therefore AB, CD, are parallel to the ordinates applied to AC (23. DEF.), and in the circle, they are at right angles to AC (18. e. 3.). Wherefore AC is the axis of the ſection (2. COR. 13.), and AB, CD, are parallel to the conjugate axis.

155. Secondly, if AB meet CD in E; I ſay, that EI, biſecting the angle AED, is an axis of the ſection.

Let X, Y, denote the focal ordinates parallel to AB, CD. The ſquare of AE is to the ſquare of CE as X is to Y; and on account of the circle, the ſquare of AE being equal to the ſquare of CE, X will be equal to Y. Wherefore, as before, it is ſhewn that EI is parallel to an axis. Join AC, meeting EI in I. Becauſe AE is equal to CE, EI is common, and the angle AEI is equal to the angle CEI, AC will be biſected at right angles by EI (4. e. 1.). Wherefore EI is a diameter of the ſection (1. COR. 18.), and is the axis itſelf, becauſe AC is ordinately applied to it at right angles.

COR. 1. If a circle meet a conic ſection or the oppoſite hyperbolas in four points, two right lines joining any two and the remaining two points, ſhall make equal angles with either axis of the ſection.

For, if parallel to each other, they are parallel to one axis, and conſequently perpendicular to the other; or if they meet each other, a parallel to one axis biſects the angle formed by their concourſe, and conſequently a parallel to the other axis drawn through the concourſe will biſect the external angle at their concourſe, viz. each of the right lines will have an equal inclination to either axis.

COR. 2. If a circle touch a conic ſection in one point, and meet it again, or meet the oppoſite hyperbolas in two other points, the right line touching the circle and ſection at the common point of contact and the right line joining the points of concourſe, as

5 alſo

also the two right lines drawn from the point of contact to the two points of concourse, shall make equal angles with either axis. FIG.

The first part of this COR. is illustrated in the same manner as the preceding COR., but the latter part is not so immediately obvious.

The same things remaining as in CASE 2., join AC, AD; I say, that they also make equal angles with either axis. 152, 153.

Draw FG parallel to CD, and touching the circle in F. This will either be AB itself, or meet it in a point G. In the first case, AO bisecting CD will be the diameter of the circle and the axis of the section. In the latter case, draw GR parallel to EI, and therefore bisecting the angle AGF. Wherefore GR will be perpendicular to AF (4. e. 1.), and consequently EI will also be perpendicular to AF, that is, because EI is parallel to an axis, AF will be parallel to the ordinates applied to that axis. But in both cases, because FG touches the circle, and CD parallel to it meets it in C, D, the arch CF will be equal to the arch DF, and the angle CAF will be equal to the angle DAF. Therefore, because AO in the one case is an axis of the section, and in the other AF is parallel to the ordinates applied to an axis, AC, AD, will make equal angles with either axis of the section.

COR. 3. Universally therefore, if a circle meet a conic section in four points, touch it in one and meet it in two more, or touch it in two points; then, considering each point of contact as a double point, any two right lines, which may be drawn to join two of the points and the remaining two, shall make equal angles with either axis of the section.

COR. 4. Conversely, if two right lines, each meeting a conic section or the opposite hyperbolas in two points, or the one touching, and the other cutting the section or sections in two points, or both touching the section or sections, do make equal angles with an axis, or what is the same thing, with a parallel to an axis; then, in the first case, the four points shall be in a circle; in the second, the circle which passes through the three points, shall touch the section in the point of contact; in the third, a circle, whose centre

A a is

FIG. is in an axis of the ſection, will touch the ſection in both the points of contact. And, if the right lines meet each other; in the firſt caſe, the rectangles under the ſegments intercepted between their concourſe and the ſection; in the ſecond, the ſquare of the ſegment of the touching line and the rectangle under the ſegments of the cutting line; in the third, the ſegments of the touching lines, ſhall be equal between themſelves.

151. If in the firſt caſe, AB be parallel to CD, becauſe PO is an axis of the ſection to which AB, CD are ordinately applied, it is evident that a circle which paſſes through three of the points, A, B, C, will have its centre in the axis, and paſs through the other point D.

150. If AB, CD, meet in E. Becauſe they make equal angles with an axis, or with a parallel to an axis, X, Y, the focal ordinates parallel to AB, CD, will alſo make equal angles with an axis, and therefore be equal between themſelves (1. COR. 27.). But the rectangle AEB is to the rectangle CED as X is to Y (23.), therefore the rectangle AEB is equal to the rectangle CED, and conſequently the four points A, B, C, D, are in a circle.

153. If in the ſecond caſe, AB touching the ſection be parallel to CD cutting it, the axis to whoſe ordinates AB, CD, are parallel, paſſes through A, and biſects CD at right angles; therefore the circle which paſſes through A, C, D, will have its centre in the axis, and will touch AB in A. Wherefore the circle and the ſection having the ſame common tangent, will touch each other.

152. But in this caſe, if AB meet CD in E; becauſe they make equal angles with an axis, or a parallel to an axis, by the very ſame reaſoning as in CASE 1., it will appear, that X, Y, are equal between themſelves, that the ſquare of AE is equal to the rectangle CED, and therefore that the circle deſcribed through A, C, D, touches AB in A.

154. Laſtly, if AB, CD, be each parallel to each other, touch each an ellipſe or the oppoſite hyperbolas in A, C, and be each parallel to the ordinates applied to an axis, then AC, being joined, will be the axis (12.), and if a circle be deſcribed, whoſe diameter is AC, the

the right lines AB, CD, being at right angles thereto, will touch the circle alſo in A, C. FIG.

But if AB, CD, touching the ſection or ſections, meet in E, and make equal angles with an axis, then, as before, it will appear, that X, Y, are equal between themſelves, and that the ſquare of AE is equal to the ſquare of CE, or AE equal to CE. Let EI be drawn biſecting the angle AEC. Becauſe AE, CE, make equal angles with an axis, and alſo make equal angles with EI, therefore EI will be an axis of the ſection, to which AC, being joined, will be ordinately applied (18.), and will be applied at right angles. Therefore a circle, whoſe centre is in EI, and which touches AE in A, will paſs through C, and touch EC alſo in C. 155.

COR. 5. If a circle touch a conic ſection, or the oppoſite hyperbolas, in two points, the right line joining the two points of contact, is ordinately applied to an axis of the ſection, and if the common tangents meet each other, the ſegments of the touching lines intercepted between the points of contact and the concourſe ſhall be equal between themſelves.

PROP. 46.

If a circle touch a conic ſection in two points, and the right line joining the points of contact be ordinately applied to the focal axis, the circle ſhall fall wholly within the ſection, but if the right line joining the points of contact be ordinately applied to the conjugate axis, the circle ſhall fall wholly without the ſection.

If a circle touch an ellipſe in the points A, C, and the common tangents AB, CD, at the points A, C, be parallel between them- 154.

 ſelves,

FIG. ſelves, AC, being joined, is both a diameter of the circle and an axis of the ſection. Wherefore the centre of the circle will be the centre of the ellipſe alſo, and if this axis be the axis ſecunda, the circumference of the circle will in every point, except in A, C, be at a leſs diſtance from the centre than the circumference of the ellipſe; but if this axis be the focal or tranſverſe axis, the circumference of the circle will in every point, except in A, C, be at a greater diſtance from the centre than the circumference of the ellipſe (5. COR. 28.). Therefore in the former caſe, the circle falls wholly within, in the latter caſe, wholly without the ellipſe.

156. If a circle touch an hyperbola in A, C, and AB, CD, the common tangents at A, C, be parallel, AC as before, will be at the ſame time the diameter of the circle, and an axis of the ſection. But ſince the points A, C, are in the oppoſite hyperbolas, AC will be the focal axis, wherefore becauſe AC is the leaſt diameter which meets the oppoſite hyperbolas (6. COR. 28.), by the ſame reaſoning it is ſhewn that the circle falls wholly without the hyperbolas.

157, 158, 159. If the common tangents AB, CD, be not parallel, let them meet in E. I ſay, that accordingly as AC, which joins the points of contact, is ordinately applied to the focal axis, or to its conjugate axis, the circle ſhall fall wholly within, or wholly without the ſection.

157, 158. Draw EI biſecting the angle AEC. EI is an axis of the ſection (45.), and if AC be ordinately applied to the focal axis, EI is the focal axis. Let it meet the ſection in G, and draw GH parallel to AC, meeting AE in H, and join HP. Becauſe GH is parallel to the ordinates applied to the axis EI, it touches the ſection in the vertex G; therefore the ſquare of HG is to the ſquare of AH as the focal ordinate parallel to HG is to the focal ordinate parallel to AH (23.). But the focal ordinate parallel to HG is leſs than the focal ordinate parallel to AH (1. COR. 28.), therefore HG is leſs than AH. But becauſe the angles at G, A, are right, and PH is a common hypotheneuſe, the ſquares of PG, GH together, are equal to the ſquares of AP, AH together.

gether. Wherefore HG being less than AH, AP will be less than PG. The circle therefore falls within the section between the terms A, C; and because they meet in no other point than A, C (44.), the circle falls entirely within the section from A to C. And in the parabola and hyperbola it meets the axis again in a point within the section, therefore the circle falls entirely in its whole extent within each of these sections. While in the ellipse, if through the other vertex G be drawn GH to touch the section in G, and meet AE in H, it will by the same reasoning be proved that AP is less than PG. Wherefore in the ellipse also, the circle meets the axis again within the section, and consequently falls entirely within the section. FIG.

If the right line AC be ordinately applied to the conjugate axis of the ellipse, then every thing being constructed alike, the reasoning will be the same, but EI being now the axis secundus, GH will be greater than AH, GH greater than AH, and AP greater than either PG or PG. Wherefore the circle will cut the axis in two points without the section, and by a parity of reasoning will appear to fall entirely without the section. 159.

In the same manner, when the circle touches the opposite hyperbolas, and AC is ordinately applied to the axis secundus EI, it might be shewn that the circle falls without each hyperbola. But this is immediately manifest, because the circle must be wholly comprehended within the angle AEC of the tangents AE, CE, while the hyperbolas are wholly situate on the parts of the tangents AE, CE, opposite to the angle AEC. The circle therefore falls wholly without each hyperbola. 160.

COR. 1. If a right line touch a conic section, and a perpendicular thereto at the point of contact meet the focal axis or either axis of the hyperbola, this perpendicular shall be the least right line which can be drawn from the point of concourse with the axis to meet the section, but if the perpendicular meet the axis secundus of the ellipse, it shall be the greatest right line which can be drawn from the concourse with the axis to meet the section.

FIG. 157, 158. The ſame things remaining, and P being the concourſe of the perpendicular with the focal axis; I ſay, that PA ſhall be the leaſt right line which can be drawn from P to meet the ſection.

Draw AC ordinately applied to the focal axis, meeting the ſection again in C, and CE touching the ſection again in C. AC, CE, will make equal angles with either axis, for they will either be both parallel to the axis, or the axis will be the diameter drawn through their concourſe (44.), and therefore will perpendicularly biſect the baſe AC, as alſo the angle AEC. Wherefore PC being joined, will be equal to PA, and the angle PCE be equal to the angle PAE, viz. each will be a right angle. A circle therefore deſcribed round P with the diſtance PA or PC, will touch PA, PC, and conſequently the ſection, in the points A, C (9. DEF.). But this circle falls entirely within the ſection, therefore PA or PC is the leaſt right line which can be drawn from the point P to meet the ſection.

160. In the ſame manner would the COR. be demonſtrated, if the perpendicular meet the axis ſecundus of the hyperbola or ellipſe. For the perpendicular will be the ſemidiameter of the circle, and therefore in the hyperbola be the leaſt right line which can be drawn from the point P to meet the ſection, becauſe the circle will fall wholly without each hyperbola; while in the ellipſe it will be the greateſt, becauſe the circle falls wholly without the ſection.

COR. 2. Hence from a point given in the focal axis of a conic ſection a right line may be drawn, which ſhall be the leaſt that can be drawn from the point to meet the ſection.

157. Suppoſe it to be done; viz. that P is the point given in the focal axis PG, and PA is drawn the leaſt right line which can be drawn from P to meet the ſection.

Let PG meet the directrix in R, the ſection in G, and F be the focus. Draw AE touching the ſection in A, AI ordinately applied to the axis, meeting it in I, and join PA. Becauſe PA is the leaſt right line which can be drawn from P to meet the ſection, it will be perpendicular to AE (by the preceding COR.); therefore

therefore PF is to IR in the duplicate ratio of FG to GR (30.). FIG.
But PF, FG, GR, are each given, therefore IR is given in magnitude, and the point R being given, the point I, and consequently the point A, are given. But it is necessary that IR be not less than GR, as otherwise the point A would vanish, and the problem be impossible. Wherefore let the point P be investigated, when IR is equal to GR, viz. when the points I, G, coincide, which is the extreme limit. Let p be this point in the axis. Then pF is to GR in the duplicate ratio of FG to GR, viz. pF is to FG as FG to GR, and (componendo) pG is to FR as FG to GR. But the semi-latus rectum of the focal axis is to FR as FG to GR (1. Def.), therefore pG is equal to the semi-latus rectum, and consequently if the distance of P from the vertex G be not less than the semi-latus rectum of the focal axis, the problem is possible and is done; and if it be equal thereto, the vertex G itself will be the point.

Cor. 3. Hence also from a point given in the axis secundus of an hyperbola or ellipse a right line may be drawn, which in the hyperbola shall be the least, but in the ellipse shall be the greatest line, which can be drawn from the point to meet the section.

Let P be the point given in OG the axis secundus of the hyperbola or ellipse, and suppose PA to be drawn to the point A in either section, which in the hyperbola is the least and in the ellipse is the greatest, that can be drawn from P to the section. Then O being the centre of the section, OG the semi-axis secundus, G its vertex, and AI ordinately applied to it; let L represent the semi-latus rectum of the axis. Because PA will be perpendicular to AE touching the section in A, PI is to OI as L is to OG (3. Cor. 30.). But the points O, P, are given, and the right lines L, OG, are given in magnitude; therefore the point I is given, and consequently the point A. But in the ellipse it is necessary that OI be not greater than OG, as otherwise the point A would vanish. Wherefore let OI be equal to OG, which is the extreme limit, viz. let the points I, G, coincide, and let p, in this case, be the point

159, 160.

159.

FIG. point given in the axis. Then pG is to OG as L is to OG; wherefore pG is equal to L, when the problem is barely possible, and G is the point required. If OI be less than OG, because PI is to OI as L is to OG, PI will be less than L, but by division of the last ratio, PI is to OP as L or pG is to pO; wherefore when PI is less than L, OP is less than Op, and the point P is nearer to the centre than p is.

In the hyperbola there will be no limit, and the problem is always possible.

PROP. 47.

If a circle touch a conic section or the opposite hyperbolas in two points, then if the section be a parabola, the semidiameter of the circle shall be a mean proportional between the semi-latus rectum of the axis and the semi-latus rectum of the diameter drawn through either of the points of contact.—But in the ellipse and hyperbola, if the circle fall within the section, the semidiameter of the circle shall be to the semidiameter of the section parallel to the common tangent at either of the points of contact, as the semi-axis secundus is to the semi-axis transverse; and if the circle fall without the section or sections, the semidiameter of the circle shall be to the semidiameter of the section parallel to the tangent as the semi-axis transverse is to the semi-axis secundus.

Let

Let a circle touch a conic section in D, D, and DE be drawn touching the circle and section in either point D, also DP be drawn perpendicular to DE, meeting an axis in P. FIG. 161, 162, 163.

If the section be a parabola, and through D be drawn the diameter DC; I say, that PD is a mean proportional between the semi-latus rectum of DC, and the semi-latus rectum of the axis. 161.

Let DC and the axis meet the directrix in C, L, DE meet the axis in E, and F being the focus, A the vertex of the axis, join FD; also draw FG parallel to DE, meeting DC in G, and joining Dd, let it meet the axis in I. Dd is ordinately applied to the axis (5. COR. 45.); wherefore PD is a mean proportional between EP, PI (COR. 8. e. 6.). But FD is equal (to DC (6. COR. 1.). viz. to DG (5. COR. 17.) viz.) to FE. Wherefore EDP being a right angle, and FE being equal to FD, F is the centre of the circle described round E, D, P (31. e. 3.), and EP is double to FD or DC, viz. EP is equal to the semi-latus rectum of the diameter DC (5. COR. 17.). Also IP is equal to the semi-latus rectum of the axis (1. COR. 30.); therefore PD is a mean proportional between the semi-latus rectum of the diameter DC, and the semi-latus rectum of the axis.

Again in the ellipse and hyperbola, let the circle fall within the section, then the centre P of the circle is in the focal axis, and if CO the semidiameter of the section parallel to DE be drawn, and CA, Ca, be the semi-axes, of which CA is the transverse; I say, that PD is to CO as aC is to AC. 162, 163.

Let DE meet aC in e, DP meet aC in p, and other things remaining the same, let L represent the semi-latus rectum of AC. Because of the equiangular triangles Dep, DEP, De is to pD as PD is to DE, and because AC, aC, are conjugate diameters, De is to CO as CO is to DE (8. COR. 24.), therefore, ex æquo perturbate, CO is a mean proportional between PD, pD, and consequently PD is to pD in the duplicate ratio of PD to CO. But on account of the parallels DI, pC, PD is to pD (as PI is to IC, viz. as L is to AC (3. COR. 30.), viz.) in the duplicate ratio

FIG. tio of aC to AC (25. DEF.). Wherefore PD is to CO as aC is to AC.

And by the same reasoning, when the circle falls without the section or sections, it would be shewn that pD is to CO as AC is to aC.

COR. 1. If two circles touch the same ellipse or hyperbolas each in two points, having one of the points of contact common to each circle, and therefore fall the one within, the other without the section or sections, the semidiameter of the section parallel to the tangent at either of the points of contact shall be a mean proportional between the semidiameters of the circle.

162, 163. This has appeared in the demonstration, viz. that PD is to CO as CO is to pD.

COR. 2. The semidiameter of the inner circle is to the semidiameter of the outward circle as the semi-latus rectum of the axis transverse is to the semi-axis transverse, or as the semi-axis secundus is to the semi-latus rectum of the axis secundus.

COR. 3. Hence it follows, that in an ellipse or hyperbola, the parallelogram formed under two conjugate diameters or semidiameters, and in the angle of the conjugates, is equal to the parallelogram formed in like manner under two other conjugate diameters or semidiameters.

The same things remaining, join CD; then CD, CO, are conjugate semidiameters, and AC, Ca, are conjugate semi-axes. Compleat the parallelograms DCOH, $ACaL$; I say, that these parallelograms are equal between themselves.

Draw CG parallel to PD, viz. perpendicular to the tangent ED, meeting it in G, and compleat the rectangle GCON. By this PROP. PD is to CO as aC is to AC, and PD is to aC as aC is to CG (10. COR. 24.), therefore, ex æquo, CO is to aC as AC is to CG. Wherefore the rectangle OCG is equal to the rectangle ACa, viz. the rectangle OG is equal to the rectangle Aa. But because the rectangle OG and the parallelogram OD lie between the same parallels, and stand upon the same base CO, they are equal between themselves (35. e. 1.), therefore the parallelogram OD is equal to the rectangle Aa.

PROP.

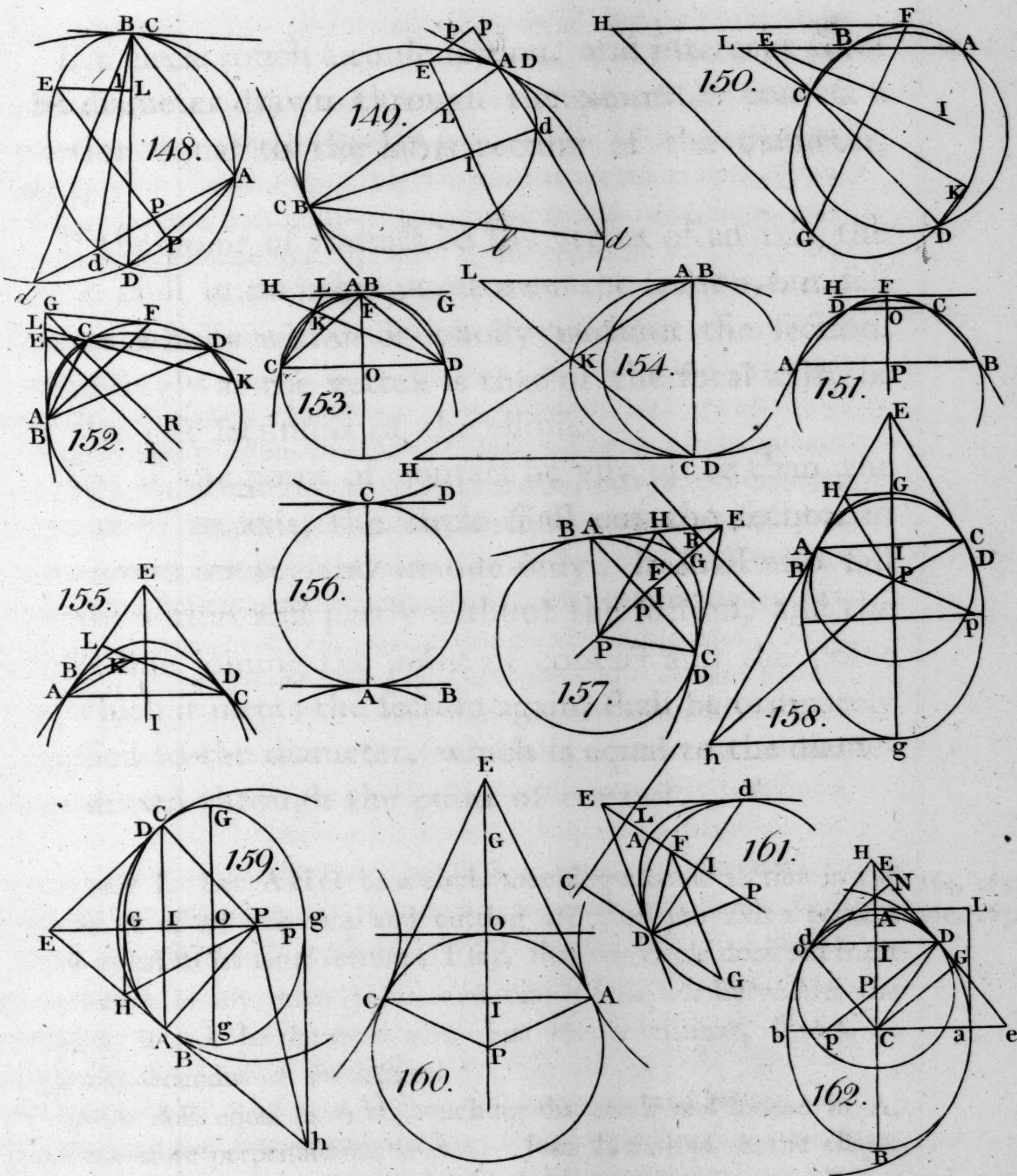
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FIG.

PROP. 48.

If a circle touch a conic ſection, and intercept from the diameter drawn through the point of contact a portion equal to the latus rectum of the diameter, then

If the point of contact be the vertex of an axis, the circle ſhall in no other point meet the ſection, but fall either wholly within or wholly without the ſection, accordingly as the vertex is that of the focal axis, or of the axis ſecundus of the ellipſe.

But if the point of contact be any other than the vertex of an axis, the circle ſhall cut the ſection in one point more, and in one only; it ſhall alſo fall partly within and partly without the ſection, and the right line joining the point of contact and the point in which it meets the ſection again, ſhall be ordinately applied to the diameter, which is equal to the diameter drawn through the point of contact.

PART I. Let AHG be a circle touching a conic ſection in the vertex A of an axis A a and cutting off from the axis a portion AH equal to its latus rectum; I ſay, that the circle does not meet the ſection in any other point, and that it falls wholly within the ſection, if A a be the focal axis, but wholly without, if A a be the axis ſecundus of the ſection. 164, 165, 166, 167.

Draw AE equal to AH, touching the circle and ſection in A, and therefore perpendicular to A a. Join HE, and in the ellipſe and hyperbola join a E, but in the parabola draw E a parallel to the

FIG. the axis. Through any point G in the circle draw a parallel to AE, meeting the section in D, Aa in B, HE in I, and aE in L. Because AE is equal to AH, BI will be equal to HB, and the rectangle ABI equal to the rectangle ABH. Also when Aa is the focal axis, because in the ellipse AH is less than Aa (25. DEF.), in the hyperbola the point a is on the parts of A opposite to H, B, and in the parabola Ea is parallel to AH; therefore BL will be greater than BI, and the rectangle ABL greater than the rectangle ABI; while in the case of the axis secundus of the ellipse, AH being greater than Aa, BI will be greater than BL, and the rectangle ABI greater than the rectangle ABL. But the rectangle ABL is equal to the square of BD (39.), and the rectangle ABI or ABH is equal to the square of BG (35. & 3. e. 3.). Therefore when the axis is the focal axis, the square of BD is greater than the square of BG, viz. BD is greater than BG, and the point G is within the section; while in the case of the axis secundus, BG will be greater than BD, and the point G be without the section. For the same reason will every point in the circle be in the one case within, in the other without the section. The circle therefore accordingly falls wholly within or wholly without the section.

', 169. PART II. Let a circle touch a conic section in a point A, which is not a vertex of an axis, and let it intercept from the diameter Aa drawn through the point of contact A a portion AH equal to its latus rectum; I say, that this circle shall meet the section in one point more, and that the right line joining this point and the point of contact A shall be an ordinate to that diameter, which is equal to Aa.*

* By two equal diameters of a conic section are understood two diameters whose parameters are equal. This will be when the vertices of the diameters are equidistant from the directrix, or when the right line joining their vertices is ordinately applied to the axis (5. COR. 17.); or in the ellipse and hyperbola, it will be when the diameters make equal angles with an axis (2. COR. 27.).

First,

First, Let the section be the parabola. Draw AD ordinate to the axis, meeting the section again in D, and draw the diameter Dd, which will be the diameter equal to Aa. Draw AF ordinately applied to Dd, meeting it in E, and the section again in F. Through A draw IAi touching the section and circle in A, and parallel to AI draw DQ, EG, meeting Aa in Q, G. AQ is equal to DE (6. Cor. 27.), viz. to QG; therefore AG is double to DE, and the rectangle HAG double to the rectangle under HA, DE. But because the diameters are equal, the latus rectum of Dd is equal to AH, and therefore the rectangle HAG is double to the square of the semi-ordinate AE (1. Cor. 39.). Again, because AF is double to AE, the rectangle FAE is also double to the square of AE; therefore the rectangle HAG is equal to the rectangle FAE, and consequently the four points E, F, H, G, are in a circle (36. e. 3.). The angle AHF is therefore equal to the external angle AEG (Cor. 22. e. 3.), viz. to the alternate angle FAI. Wherefore the point F is in the circle touching the section in A (Cor. 32. e. 3.), viz. the circle does meet the section again in F, and AF is ordinately applied to the diameter Dd, equal to Aa. FIG. 168.

Secondly, Let the section be an ellipse or hyperbola. Every thing remaining as in the parabola, draw also the diameters Kk, Ss, conjugate to Aa, Dd, the former meeting AF in G, and the latter meeting AI in L. Draw AQ ordinate to Ss, meeting it in Q, and let C be the centre of the section. Because the diameters Aa, Dd, are equal between themselves, their conjugates Kk, Ss, will also be equal; and because AH is the latus rectum of Aa, the rectangle HAa will be equal to the square of Kk or Ss (25. Def.), viz. will be quadruple to the square of SC. But, Aa being bisected in C, the rectangle HAa will be double to the rectangle HAC. Therefore the rectangle HAC is double to the square of SC. Again, because GA, AE, are severally equal to LC, CQ, the rectangle GAE is equal (to the rectangle LCQ, viz.) to the square of SC (9. Cor. 24.). But AF is double to AE, 169.

FIG. AE, therefore the rectangle FAG is double to the rectangle GAE and consequently is double to the square of SC. Wherefore the rectangle HAC is equal to the rectangle FAG, and the four points C, H, F, G, are in a circle. As before therefore, the angle AHF is equal (to the external angle AGC, viz.) to the alternate angle FAI; and consequently the point F is in the circle touching the section in A. The circle therefore does meet the ellipse or hyperbola again in a point F, and AF is ordinately applied to the diameter Dd, equal to Aa.

I say also, that the circle does not meet the section again in any other point than F. For if possible, let them meet again in R, and join AR. Then AF, AR, make the same angles with either axis (2. Cor. 45.). But AF is parallel to DI, and DI, AI, make equal angles with each axis (3. Cor. 45.); therefore AF, AR, AI, drawn from the same point A, make equal angles with an axis, that is, with the same right line, which is absurd (16. e. 1.). The circle therefore does not meet the section again in any other point than F.

Lastly, because the circle touches the section in A, and meets it in another point F, but does not touch it in F, because FA is not perpendicular to an axis; it must necessarily fall between the points A, F, on each side, partly without, and partly within the section.

Cor. 1. If in the focal axis of a conic section, or the axis secundus of the hyperbola, a point be assumed, whose distance from the vertex is either equal to, or less than the semiparameter of the axis, this distance shall be the least of all the right lines which can be drawn from the point to the section; but if the point be taken in the axis secundus of the ellipse, and its distance from the vertex be equal to or greater than the semiparameter of the axis, this distance shall be greater than any other right line which can be drawn from the point to the section.

164, 165, 166. The same things remaining, PA is, in the first case, the least right line, which can be drawn from the point P to the section, because

because the section touching the circle in A falls in every point without. If pA therefore be less than PA, the circle described round p with the distance pA will fall entirely within the former circle, and therefore pA will be less than any other right line which can be drawn from p to the circumference of this outer circle, and therefore much less than any other right line drawn from p to the section. FIG.

By the same reasoning it is shewn, if PA, pA, be assumed in the axis secundus of the ellipse, and either equal to, or greater than, the semi-parameter of the axis, that PA, pA, are greater than any other right line which can be drawn from P, p, to meet the section.

COR. 2. Hence therefore, if a circle whose centre is in the focal axis of a conic section, or in the axis secundus of the hyperbola, and which passes through the vertex of the axis, meet the section again, the semidiameter of this circle must be greater than the semi-parameter of the axis. But if the centre of the circle be in the axis secundus of the ellipse, the semidiameter of the circle must be less than the semi-parameter of the axis.

For if not, the distance of the centre from the vertex would, in the one case, be the least, and in the other, the greatest right line which can be drawn from the centre to the section; which is contrary to the hypothesis, because there is another point in the section, which is supposed to be at an equal distance from the centre.

COR. 3. If a point be assumed in the focal axis of a conic section, or in the axis secundus of the hyperbola, whose distance from the vertex is greater than the semi-parameter of the axis, a right line may be drawn therefrom, on either side of the axis, to the section, which shall be less than any other right line drawn from the point to the section. But if the point be taken in the axis secundus of the ellipse, and its distance from the vertex be less than the semi-parameter of the axis, a right line may be drawn therefrom to the section, on either side of the axis, which shall be greater than any other right line drawn from the point to the section.

If

FIG. 172, 173, 174. If the axis be that of the parabola, or the focal axis of the ellipse, or either axis of the hyperbola, and P be the given point therein, A the vertex, and PA be greater than PA the semi-latus rectum of the axis; in the parabola, take PQ towards the vertex A, equal to PA, but in the ellipse and hyperbola, take PQ such that C being the centre of the section, PQ shall be to CQ as PA is to AC. If QO be ordinately applied to AC, meeting the section in O, because PQ is to CQ as PA is to AC, then dividendo or componendo, PC is to CQ as PC is to AC; therefore PA being greater than PA, PC will be less, in the ellipse, and greater, in the hyperbola, than PC; and consequently CQ will be less or greater than AC. Wherefore in both sections, the point Q will be within the section. But in the parabola, PA being greater than PA, viz. than PQ, the point Q will be within the section. Wherefore in each of the three, QO will meet the section, on each side of the axis, in O, o. Join PO or Po, each of which shall be the right line required. For the circle described round the centre P with the distance PO or Po shall touch the section in O, o, and fall entirely within the section, or entirely without either hyperbola, when P is in the axis secundus, (1. & 2. Cor. 30. & 44.). Therefore PO, Po, towards either part of the axis, are each of them less than any right line which can be drawn from P to meet the section.

In the same manner it may be shewn, if the axis be the secundus of the ellipse, that a right line may be drawn from the point given therein to meet the section, on either side of the axis, which shall be greater than any drawn therefrom to the section.

PROP. 49.

If a circle touch a conic section, and intercept from the diameter drawn from the point of contact a segment equal to its parameter, the circle shall have the same

same curvature, which the section has in the point of contact. FIG.

Case I. When the circle touches the section in the vertex of an axis.

Let the circle AGH touch a conic section in the vertex A of the focal axis, or of the axis secundus of the ellipse, and intercept from the axis a segment AH equal to its parameter; I say, that the circle and the section shall have the same curvature in the point A. 172, 173. 174.

For if it be possible that a circle may be described, which passes between the circle AGH and the section, it must touch both in the point A, viz. it must have its centre in the axis; because a circle, whose centre is not in the axis, would cut the common tangent in A, and consequently fall partly within, and partly without the circle and section; and moreover this circle must, in the case of the focal axis, intercept a diameter greater, and in the case of the axis secundus of the ellipse a diameter less, than AH, because otherwise the circle would fall wholly within or wholly without the circle AGH, and therefore not pass in any part between the circle and section.

Wherefore let a circle be in each case described, which having its centre in the axis, cuts off from the axis a segment Ah greater or less than AH. Bisect AH, Ah, in P, p. Because pA is, in the one case greater and in the other less than PA, a right line pO may be drawn to meet the section, on either side of the axis, which in the one case shall be less, in the other greater than pA (3. Cor. 48.). Let pO, po, be so drawn, meeting the circle Agh in R, r. Because pO, po, are in the one case less, in the other greater, than pA, viz. than pR or pr, the points O, o, fall accordingly within or without the circle Agh, viz. the circle Agh is in the points R, r, without or within the section, and therefore the whole circumference RAr must fall entirely without or within the section. In neither case, therefore, can a circle be described which passes

 between

FIG. between the section and circle AGH, and consequently the circle AGH has the same curvature with the section in the point A.

Case 2. When the circle touches the section in any other point than the vertex of an axis.

170, 171. Let the circle AFH touch a conic section in A, which is not the vertex of an axis, and every thing remain as in the preceding Prop. Draw AD ordinately applied to the focal axis, and in the ellipse and hyperbola draw also Ad ordinately applied to the axis secundus. If a circle can be described, which passes between the circle AFH and the section, it must touch the section in A, and therefore must be a greater circle than that which would touch the section in A, D, and must be a less circle than that which would touch the circle in A, d; because the former of these circles, and much more every less circle falls entirely within the section, while the latter, and much more every greater circle falls entirely without the section (46.). Let therefore a circle be described, which touching the section in A, is greater than the one which would touch the section in A, D, and is less than the one which would touch the section in A, d. This circle must meet the section again, because on the part of D it falls without the section, and on the part of d within the section, while in the case of the parabola, the circle falling on the part of D without the section, and the parabola being indefinite in its extent, the circle in returning to A again must cut the section. Let the circle therefore meet the section again in O. Draw OT parallel to AF, viz. to the right line DI, which touches the section in the point D, and let OT meet the section again in T, AD in Z, and AI in V. The rectangle OVT is equal to the square of ZV (31.). But because DI is equal to AI (3. Cor. 27.), therefore ZV will be equal to AV, and consequently the rectangle OVT is equal to the square of AV. Wherefore the point T is also in the circle, viz. the circle meets the section again in T. Wherefore touching the section in A, and meeting it in the two points O, T, it does not meet it in

any

any other point (44.). Now if the points O, T, fall on the same side of AF with D, because towards D the circle falls without the section, it must from A to T fall entirely within the section, from T to O without the section, and from O to A fall again entirely within the section. And in like manner, if the points O, T, fall on the parts of A opposite to D, viz. towards D, because it meets the section in no other point than A, O, T, and on the parts towards O falls without the section, it must from A to O fall entirely without the section, from O to T fall within, and from T to A fall again entirely without the section. Wherefore every circle that can be drawn to touch the section in A, falls on each side of A, either wholly within or wholly without the section, and therefore cannot pass between the section and the circle AFH, because the circle AFH falls on one side of A within, and on the other without the section (48.). The circle AFH is therefore of the same curvature with the section in the point A (26. Def.).

PROP. 50.

If a circle, touching a conic section in any other point than the vertex of an axis, have the same curvature with the section in that point.

Then in the parabola, the parameter of the axis shall be to the parameter of the diameter drawn through the point of contact in the duplicate proportion of the same parameter to the diameter of the circle of equal curvature.

But in the ellipse and hyperbola, the diameter of the circle of equal curvature shall be to the diameter conjugate to the one drawn through the point of

 contact,

FIG. **contact, as the square of this conjugate diameter is to the rectangle under the parameters of the axes, or to the rectangle under the axes themselves.**

170, 171. The same things remaining as in Prop. 48. and 49., draw AX perpendicular to AI, the common tangent at the point A, meeting the circle of equal curvature again in X, AX will be the diameter of this circle. Bisect AX in P, which will be the centre of the circle, and draw PN perpendicular to AH, which will bisect it
171. in N (3. e. 3.). Then Part 1. In the parabola, if AX meet the axis in p, and AD meet it in B, Bp will be equal to the semiparameter of the axis (1. Cor. 30.), and if Bp be produced to b, so that pb be equal to Bp, the whole Bb will be equal to the parameter. I say then that Bb shall be to AH in the duplicate ratio of AH to AX.

Because Ap is a mean proportional between Bp, AN (47.), Bp is to AN in the duplicate proportion of Bp to Ap; and because the triangles BAp, ANP, are equiangular, Bp is to Ap as AN is to AP; therefore ex æquo, Bp is to AN in the duplicate proportion of AN to AP, and their doubles being in the same proportion, Bb will be to AH in the duplicate proportion of AH to AX.

170. Part 2. In the ellipse and hyperbola, Kk being the diameter conjugate to Aa; I say that AX is to Kk as the square of Kk is to the rectangle under the parameters of the axes, or to the rectangle under the axes themselves.

Let AX meet Kk in Q. Because AX is perpendicular to AI, it will be perpendicular also to Kk; wherefore the angles AQC, ANP, being equal, because each right, and the angles at A being common, the triangles PAN, CAQ, are equiangular, and AP is to AN as AC is to AQ. But AN being equal to the semiparameter of AC, AN is to CK as CK is to AC (25. Def.); therefore, ex æquo perturbatè, AP is to CK (as CK is to AQ, viz.)

viz.) as the square of CK is to the rectangle under CK, AQ. FIG.
But the rectangle under CK, AQ, is equal to the parallelogram constituted under the semi-conjugate diameters AC, CK, about the angle ACK (35. e. 1.), and this parallelogram is equal to the rectangle under the semi-axes (3. Cor. 47.), or to the rectangle under the semi-parameters of the axes (25. Cor. Def.) Therefore AP is to CK as the square of CK is to the rectangle under the semi-axes, or to the rectangle under the semi-parameters of the axes; and because the doubles are in the same proportion, AX is to Kk as the square of Kk is to the rectangle under the axes, or to the rectangle under the parameters of the axes.

Cor. 1. In the parabola, if round the centre p with the distance pA a circle be described, this circle will touch the parabola in the points A, D; and the parameter of the axis will be to the parameter of the diameter of the section drawn through A as the diameter of the circle pAD is to the diameter of the circle of equal curvature.

For it has been shewn that pB is to AN as pA is to AP, and the doubling of these proportionals is the Cor. itself.

Cor. 2. In the parabola, if a mean proportional be taken between the semi-parameter of the axis and the semi-parameter of the diameter drawn through the point of contact, this mean proportional, the semi-parameter of the diameter drawn through the point of contact, and the radius of curvature shall be in continued proportion.

For Ap is a mean proportional between Bp, AN, therefore Bp is to Ap as Ap is to AN, and also Bp is to Ap as AN is to AP. Wherefore, ex æquo, Ap is to AN as AN is to AP, which is the Corollary.

Cor. 3. In the ellipse and hyperbola, the semi-diameter drawn from the point of contact is to the perpendicular drawn from the point of contact upon the conjugate as the radius of curvature is to the semi-parameter of the diameter drawn through the point of contact.

For

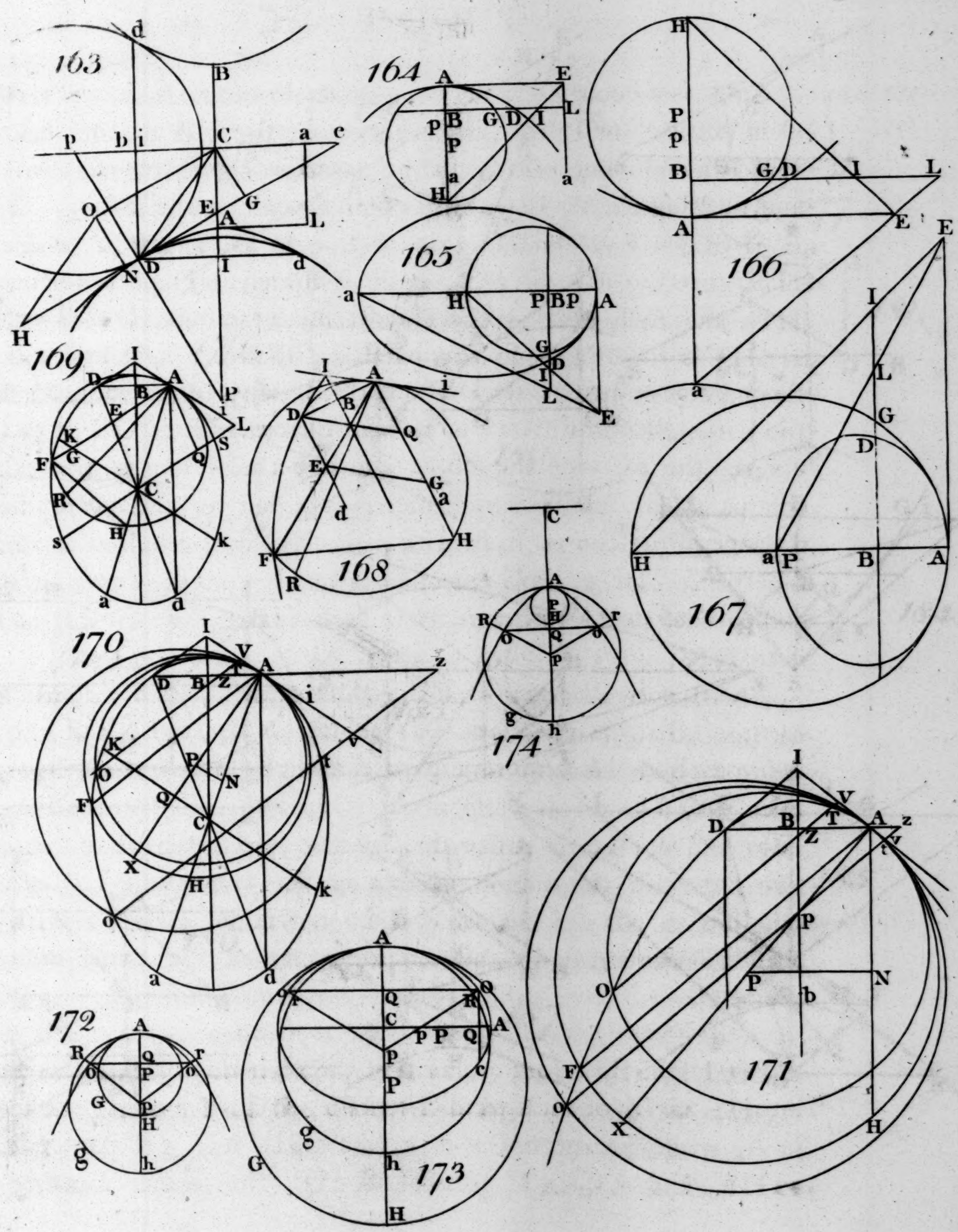
FIG. For it has appeared that AC is to AQ as AP is to AN, which is the Corollary.

PROP. 51.

188, 189. ABb is a conic ſection, Bb a diameter of it, B, b, its vertices,
190, 191. O its centre, or B the ſingle vertex in the caſe of the parabola. HE is a right line parallel to the ordinates applied to the diameter, and meeting the diameter in a point E without the ſection. From E is drawn EG touching the ſection in G, and GD parallel to HE, meeting Bb in D. At E is drawn EF perpendicular to HE, and ſuch that in the ellipſe and hyperbola, OD is to OB as GD is to EF, or in the parabola, that EF is equal to GD. Round any point C a circle is deſcribed, whoſe ſemidiameter is to the perpendicular diſtance of its centre from HE as OB is to OE, viz. in the caſes of the ellipſe and hyperbola, but in the caſe of the parabola, whoſe ſemidiameter is equal to the perpendicular diſtance of its centre from HE. Laſtly, if through C be drawn a perpendicular to HE, meeting it in H, and the circle in R, r, and parallel to Bb be drawn HS, ſuch that EF be to EB as RH is to HS.

Then, if two right lines FP, SP, revolve about the points F, S, carrying their interſection P conſtantly in the right line HE, or being each at the ſame time parallel to HE; while two other right lines FA, Sa, revolve alſo about F, S, and while the one FA follows and conſtantly paſſes through the interſection A of the right line SP with the ſection, the other Sa preſerves a conſtant paralleliſm to FA; I ſay, that the interſection a of Sa with FP ſhall be conſtantly in the circumference of the circle deſcribed round C.

Let FP, SP, be any poſition of the revolving lines, either meeting HE in P, or each parallel to HE, and SP meeting the ſection in A, while Sa meets FP in a, draw SL, AI, aK perpendicular to HE, and AV, aT, parallel to HE, the former meeting FE in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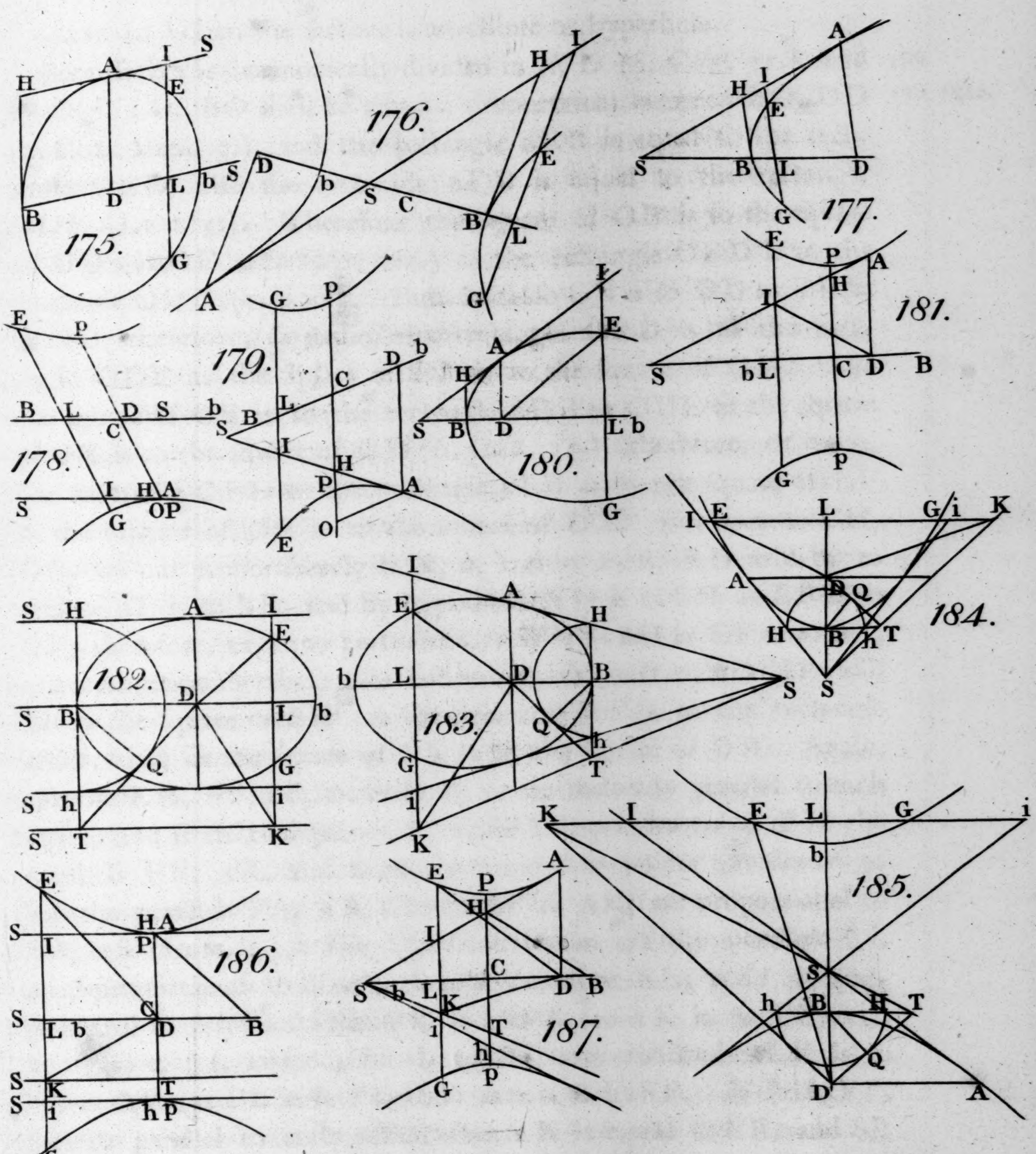
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in V, B b in Q, and the latter meeting R r in T; alſo draw XO x the diameter conjugate to BO b. Then FIG.

CASE 1. When the ſection is an ellipſe or hyperbola.

Becauſe B b is harmonically divided in E, D (3. COR. 37.), and biſected in O, BO ſhall be a mean proportional between EO, OD (2. COR. LEM. 5.), and the rectangle b EB is equal to the rectangle OED, alſo the rectangle b DB is equal to the rectangle ODE (LEM. 5.). Therefore the ſquare of OB is to the ſquare of OD (as OE is to OD, viz.) as the rectangle OED is to the rectangle ODE (1. e. 6.). But, becauſe OB is to OD as EF is to GD, therefore, ex æquo, the rectangle OED is to the rectangle ODE as the ſquare of EF is to the ſquare of GD. And the ſquare of OB is to the rectangle b DB or ODE as the ſquare of OX is to the ſquare of GD (6. COR. 23.), therefore, ex æquo, the rectangle OED or the rectangle b EB is to the ſquare of EF as the ſquare of OB is to the ſquare of OX. But becauſe CH, OE, are cut proportionally in R, b, and in r, B, RH will be to r H as b E is to BE, and by hypotheſis RH is to SH as EF is to BE, therefore, ex æquo perturbatè, r H is to SH as EF is to b E, and compounding theſe two laſt ratios, the rectangle r HR will be to the ſquare of SH (as the ſquare of EF is to the rectangle b EB, viz.) as the ſquare of OX is to the ſquare of OB. Again, becauſe EH, FP, SP, meet in P, or are mutually parallel to each other, and from two points F, a, in FP, are drawn to EH the parallels FE, a K, and from the ſame two points are drawn to SP the parallels FA, a S, therefore FE, a K, are proportional to FA, a S (LEM. 1.). For the ſame reaſon are the parallels AI, SL, proportional to FA, a S, and therefore FE, a K, are proportional to AI, SL. But PK is to PE (as a K is to FE, viz. as SL is to AI, viz.) as PL is to PI, and dividendo, LK is to EI or AV (as PL is to PI, viz.) as a K is to FE. If EH, FP, SP be parallel to each other, then a K is equal to FE, and SL to AI. Alſo becauſe FA is parallel to a S, F a S A becomes a parallelogram, wherein SA is equal to F a, but for the ſame reaſon

AS

188.
190, 191.

FIG. SA is equal to LI, and Fa to EK, therefore LI is equal to EK, and taking away or adding the common magnitude IK, the remainders or the wholes LK, EI, will be equal. Wherefore equal things being proportional to equal things, LK will, as before, be to EI or AV as aK is to FE. Moreover in both cases, on account of the similar triangles HSL, QEV, HL is to QV (as SL is to EV or AI, viz.) as aK is to FE. Wherefore, ex æquo, LK is to HL as AV is to QV, viz. LK, AV, are proportionally cut in H, Q, and HK is to AQ (as LK is to AV, viz.) as aK is to FE. The square of HK is therefore to the square of AQ as the square of aK is to the square of FE, and componendo, the square of aH is to the squares of FE, AQ, as the square of aK is to the square of FE. But aK is to FE (as HL is to QV, viz.) as SH is to EQ. Wherefore ex æquo, the square of aH is to the squares of FE, AQ, as the square of SH is to the square of EQ. Again, the rectangle bQB is to the square of AQ as the square of OB is to the square of OX (6. Cor. 23.), and it has been shewn that the rectangle bEB is to the square of EF as the square of OB is to the square of OX; therefore ex æquo, the rectangle bQB is to the square of AQ as the rectangle bEB is to the square of EF; and componendo, and in OE taking Oq equal to OQ, the rectangle qEQ will be to the squares of FE, AQ, as the rectangle bEB is to the square of FE *. But it has also been shewn that the rectangle bEB is to the square of FE as the square of SH is to the rectangle rHR. Therefore, ex æquo, the rectangle qEQ is to the squares of FE, AQ as the square of SH is to the rectangle rHR, and it has just been proved that the square of aH is to the squares of FE, AQ, as the square of SH is to the square of EQ, wherefore, ex æquo perturbatè, the rectangle rHR will be to the square of aH (as the square of EQ is to the rectangle qEQ, viz.) as EQ is to Eq, and componendo or dividendo, the rectangle rHR is to the rectangle rHR together with the square of aH, or to

* That the rectangle bQB together with the rectangle bEB is equal to the rectangle qEQ, vide Note Prop. 42.

the

the excess of the rectangle rHR and the square of aH as EQ FIG.
is to the double of EO. Lastly, it has been shewn that EQ is to SH as FE is to aK or HT, and by hypothesis, RH is to SH as FE is to EB, therefore, ex æquo perturbatè, RH is to HT as EQ is to EB, and it has also appeared that rH is to CH as EB is to EO; therefore, compounding these two last ratios, the rectangle rHR is to the rectangle CHT (as the rectangle BEQ is to the rectangle BEO, viz.) as EQ is to EO. Whence by doubling the consequents, the rectangle rHR is to double the rectangle CHT as EQ is to the double of EO. But it has just been proved that the rectangle rHR is to the rectangle rHR together with the square of aH, or to the excess of the rectangle rHR and the square of aH, as EQ is to the double of EO. Therefore the rectangle rHR together with the square of aH, or the excess of the rectangle rHR and the square of aH, is double to the rectangle CHT, and consequently the point a is in the circumference of the circle described round C (LEM. 15. & 16.).

CASE 2. When the section is a parabola.

In this case the points r, H, coincide, EH touches the circle 189.
in H or r, GD is equal to FE, and BD is equal to EB (37.). Then in the same words as in the preceding case it is proved that aK or HT is to RH as EB is to EQ, and that the square of aH is to the squares of FE, AQ, as the square of aK or HT is to the square of FE. But the square of GD is to the square of AQ as BD is to BQ, (6. COR. 22.), viz. the square of FE is to the square of AQ as EB is to BQ, and componendo, the square of FE is to the squares of FE, AQ (as EB is to EQ, viz. as HT is to RH, viz.) as the square of HT is to the rectangle RHT. Therefore, ex æquo perturbatè, the square of FE is to the square of aH as the square of FE is to the rectangle RHT. Consequently the square of aH is equal to the rectangle RHT, and therefore the point a is in the circumference of the circle described round C (31. e. 3. & COR. 8. e. 6.).

FIG. 192, 193. 194.

COR. When the diameter is the focal axis, then EF coincides with the axis, and if the point D fall in the focus; I ſay that HE ſhall be the directrix of the ſection, that the point F ſhall fall alſo in the focus, that the circle deſcribed round C ſhall be a generating circle, and that the point S ſhall fall in the centre C.

Becauſe DG, ordinately applied through the focus to the focal axis, meets the ſection in G, it is equal to the principal ſemi-parameter (5. COR. 1.), and becauſe GE touching the ſection in G, meets the axis in E, therefore E will be the point in the axis through which the directrix paſſes (6. COR. 9.), viz. EH is the directrix.

Again, becauſe D is the focus, DG the principal ſemi-parameter, and EH the directrix, therefore in the ellipſe and hyperbola, DG will be to ED as OB to EO (16. DEF. & 4. COR. DEF., viz.) as OD to OB (17. COR. DEF.) But, by hypotheſis, DG is to EF as OD is to OB; therefore EF is equal to ED. While, in the parabola, D being the focus, ED is equal to the principal ſemi-parameter (7. COR. 1.), viz. to DG; and, by hypotheſis, EF is equal to DG, therefore EF is equal to ED. Wherefore, in each caſe, the point F coincides with D, viz. it falls in the focus.

Moreover, in the ellipſe and hyperbola, becauſe CR is to CH (as OB is to EO, viz.) as DG is to ED, therefore the circle deſcribed round C is a generating circle (16. DEF.), while in the parabola every circle touching the directrix is a generating circle (6. COR. DEF.).

Laſtly, becauſe EF, EO coincide, HS, HC coincide alſo; and in the ellipſe and hyperbola, becauſe Eb is to EF as EO is to EB (17. COR. DEF.), but Eb is to EF as HS is to rH, and EO is to EB as HC is to rH; therefore HS is to rH as HC is to rH, viz. the point S falls in the centre. Alſo in the parabola, becauſe EF is double to EB (7. COR. 1.), and EB is to EF as HS is to HR; therefore HR is alſo double to HS, viz. the point S falls in the centre C.

PROP. 52. LOCUS.

Let rAR be a circle given in position and magnitude, EH a right line given in position, and S, F, being two given points, without the right line EH, let the right lines SP, FP revolve about S, F, and carry their intersection P constantly in the right line EH, or be at the same time each parallel to EH, and about the same points S, F, let the right lines Sa, FA revolve, so that while Sa constantly follows the intersection a of FP with the circle, the other FA shall preserve a constant parallelism to Sa; I say, that the point A, which is the intersection of FA with SP shall by its motion describe a CONIC SECTION. FIG. 188, 189. 190, 191.

Through C the centre of the circle and the points S, F, draw CH, SL, FE, perpendicular to HE, the former meeting the circle in the points R, r. Join SH, and draw EO parallel thereto; also join FR, and accordingly as it meets HE, or is parallel to it, draw Sp, meeting HE in the same point p, or parallel to HE. Let Sp meet EO in B, and join SR, FB. Then if FR, Sp, meet HE in p, because of the parallels, Rp is to RF (as Hp is to HE, viz.) as Sp is to SB. Wherefore SR is parallel to FB. If FR, Sp be parallel to HE, then SHEB, RHEF are parallelograms, wherein SB, RF, being each equal to HE, are equal to each other, and therefore as they are also parallel, SR will, as before, be parallel to FB. Wherefore the point B is in the section, or line described by the motion of A. For the same reason, if CH meet the circle in another point r, distinct from H, viz. if the circle do not touch HE, there will be another point b in EO, correspondent to the point r, which will also be in the section.—In this case bisect Bb in O. Because the sides of the triangle FEB are respectively parallel to the sides of the triangle RHS, EB will be to EF as SH is to HR, and for the same

 reason,

FIG. reason, E b is to EF as SH is to H r, therefore ex æquo perturbatè, EB is to E b as H r is to HR, and also, compounding the same two ratios, the rectangle b E B is to the square of EF as the square of SH is to the rectangle r HR. Wherefore r R, B b are proportionally cut in H, E, and being bisected in C, O, CH, OE will be proportionally cut in R, b, and in r, B.

Draw AI, a K, perpendicular to HE, and AV, a T, parallel to HE, meeting EF, CH, in V, T, and the latter meeting also EO in Q. In EO on the part of O opposite to Q, make O q equal to OQ, and join a H. Then by the same reasoning, and in the same words, as in the preceding, it is proved, that the square of a H is to the squares of FE, AQ, as the square of SH is to the square of EQ, that the rectangle r HR is to the double of the rectangle CHT as EQ is to the double of EO, that HT is to EB as HR is to EQ, and that HR is to FE as SH is to EB.

Case 1. When the circle does not touch HE.

188, 190. 191. Because the rectangle r HR is to twice the rectangle CHT as EQ is to twice EO; therefore componendo or dividendo, the rectangle r HR is to the square of a H (as EQ is to E q (Lem. 15, 16.)*, viz.) as the square of EQ is to the rectangle q EQ. But it has been proved, that the square of a H is to the squares of FE, AQ, as the square of SH is to the square of EQ; therefore, ex æquo perturbatè, the rectangle r HR is to the squares of FE, AQ, as the square of SH is to the rectangle q EQ. It has also been shewn, that the rectangle r HR is to the square of FE as the square of SH is to the rectangle b EB. Therefore ex æquo, the square of FE is to the rectangle b EB as the squares of FE, AQ, are to the rectangle q EQ, and dividendo, the square of FE is to the rectangle b EB as the square of AQ is to the rect-

* Twice EO together with EQ, or the excess of twice EO above EQ, accordingly as the composition or division of the ratio is required, is equal to E q. This is so obvious, as to need no illustration.

angle

angle bQB*. Through O draw OX parallel to HE, such that FIG.
the square of OX may be to the square of OB as the square of FE is to the rectangle bEB. Then, ex æquo, the square of OX is to the square of OB as the square of AQ is to the rectangle bQB. But this is a principal property of the ellipse and hyperbola, when a right line is ordinately applied from any point in either section to one of two conjugate diameters (6. COR. 23.). Therefore the line described by the motion of the point A is an ellipse or hyperbola, when the circle does not touch HE, and OB, OX are conjugate semidiameters.

CASE 2. When the circle touches HE.

Because it has been shewn, that HT is to EB as HR is to 189.
EQ, therefore the rectangle RHT is to the rectangle BEQ as the square of HR is to the square of EQ (1. e. 6.). But it has also been shewn, that in this case as well as in the first, the square of aH is to the squares of FE, AQ, as the square of SH is to the square of EQ; and because in this case, the rectangle RHT is equal to the square of aH (31. e. 3. & COR. 8. e. 6.); therefore, ex æquo perturbatè, the square of HR is to the squares of FE, AQ, as the square of SH is to the rectangle BEQ. It has also appeared, that HR is to FE as SH is to EB, therefore, ex æquo, the square of FE is to the squares of FE, AQ (as the square of EB is to the rectangle BEQ, viz.) as EB is to EQ; and, dividendo, the square of FE is to the square of AQ (as EB is to BQ, viz.) as the square of EB is to the rectangle under EB, BQ. Assume the right line L a third proportional to EB, FE; then the square of FE is to the square of EB (as L is to EB, viz. as the rectangle under L, BQ is to the rectangle EBQ. Therefore the rectangle under L, BQ is equal to the square of AQ, which is a principal property of the parabola, respecting an ordinate applied to a diameter (1. COR. 41.). Therefore the line described by the

* That the excess of the rectangle qEQ above the rectangle bEB is equal to the rectangle bQB, vide NOTE, PROP. 42.

motion

FIG. motion of the point A is a parabola, when the circle touches HE, of which EB is a diameter, L its latus rectum, and HE is parallel to the ordinates applied to EB.

Wherefore, universally the line described by the motion of the point A is a conic section.

Cor. 1. As the circle falls entirely without the right line HE, cuts it, or touches it, or as the perpendicular distance of the centre of the circle from HE is greater, less than, or equal to the semidiameter of the circle, the section shall be an ellipse, hyperbola, or parabola.

In each case, the line described is a conic section, but in the first case, when the circle falls entirely without HE, as the intersection A with the circle is constantly on the same side of HE, so will the intersection A with the section always fall on one and the same side of HE, and while the point A moves through the whole circumference of the circle, and returns upon itself, the point A will move on one and the same side of HE, and performing an entire revolution, return to its first position. The section is therefore a line enclosing a space round the centre O, and as it has been proved to have a principal property belonging to the ellipse, the section is an ellipse.

But in the second case, when the circle cuts the right line HE, and is divided thereby into two segments, as the point A is traversing the circumference of one of these segments, a section is described by A, which is situated on one side of HE, but when A traverses the other segment, the point A describes another section, which is situated towards the other side of HE, nor are these two sections one continued section, but separated by HE, because no point in HE can be in either section, from the very nature of the genesis. As each of these sections therefore has been proved to have a principal property of the hyperbola, the sections are two opposite hyperbolas.

In the third case, when the circle touches HE, the point A will be always situated one and the same side of HE. The section therefore

therefore will be one, and as in this case it has been proved to have FIG.
a principal property of the parabola, the section will be a parabola.

COR. 2. As the point S is in, within, or without the circle, the point F shall be in, within, or without the section.

Through S draw any right line meeting the circle in the points 195.
A, N. There will be two points A, N, in the section answering to A, N, in the circle; and FA, FN, being drawn, will be respectively parallel to SA, SN, viz. FAN will be one right line parallel to SAN. Also if the right lines revolving about S, F, by whose motion the section is described, be drawn, it will appear, that in what order the points S, A, N, are, in the same order are the points F, A, N; viz. that if the point S be within or without the points A, N, or coincide with one of them, as A, the point F shall accordingly be within or without the points A, N, or coincide with A. Therefore as a right line can neither meet the section or circle in more than two points, the point F will be in, within, or without the section, accordingly as the point S is in, within, or without the circle.

COR. 3. Hence therefore if the point F fall in the section, the 196, 197,
right line BF, and in the ellipse and hyperbola, BF also shall be 198.
ordinately applied to an axis of the section.

For the point S then falls also in the circumference of the circle, and SR being parallel to BF, SR to BF, and SH to BE, the angle RSR is equal to the angle BFB, and the angle RSH to the angle EBF. But in the ellipse and hyperbola the angle RSR, and in the case of the parabola, the angle RSH, is a right angle (31. e. 3.). Therefore in the one case, the angle BFB, and in the other the angle EBF, is a right angle. Draw the diameter Ou bisecting BF in M. Then in the former case, because BB is bisected in O, and BF in M, OM will be parallel to BF, and therefore perpendicular to BF. In the second case, because all the diameters of a parabola are parallel to each other, Ou is parallel to EB, and therefore perpendicular to BF. But in each case, because BF is bisected by the diameter Ou, it is ordinately applied

to

FIG. to Ou (1. COR. 15.), and because it is bisected at right angles, Ou is an axis of the section.

For the same reason is bF ordinately applied to the other axis.

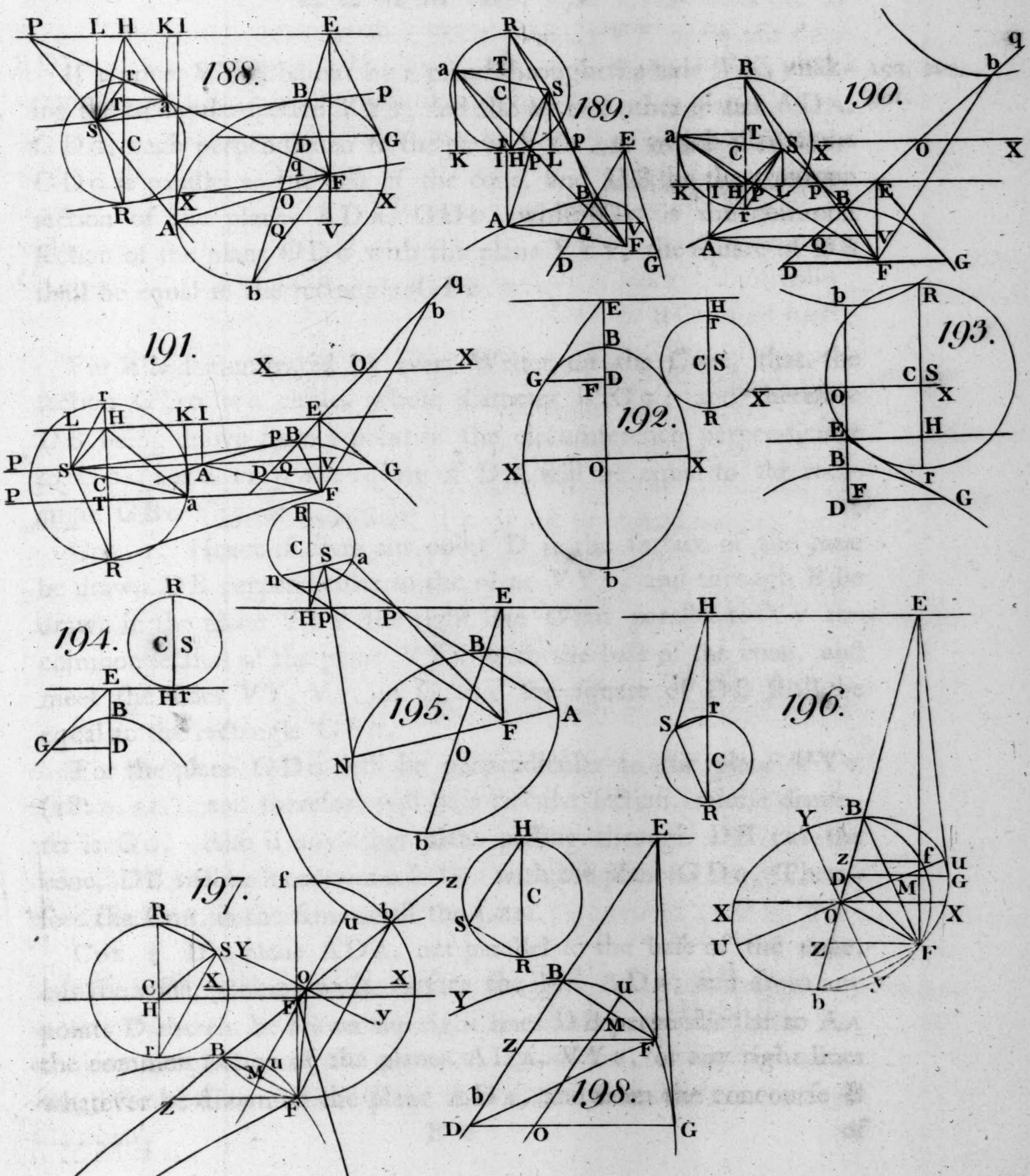
COR. 4. Hence therefore, every thing remaining as in the preceding COR., because in the ellipse and hyperbola, OD is to OB as DG is to EF, and in the parabola, DG is equal to EF; therefore if from the vertex B of a diameter Bb of a conic section an ordinate BF be applied to an axis of the section, and FE perpendicular to the ordinates applied to Bb meet Bb in E, also from E be drawn EG touching the section in G, and GD ordinately applied to Bb meet it in D; then in the ellipse and hyperbola, OD shall be to OB as DG is to EF, and in the parabola, DG shall be equal to EF. This conclusion is the same with the 1. & 3. COR. of PROP. 42.

PROP. 53.

205, 206, 207. If AD be a conic section, F its focus, XX the directrix, FE the focal axis meeting the directrix in E, the section in A, and through A be drawn any right line AH, wherein is taken AH equal to AF, and EH be joined. If through any point D in the section be drawn DB parallel to the directrix, meeting the axis in B, and BG parallel to EH meet AH in G; I say, that FD, being joined, shall be equal to HG.

Draw DI perpendicular to the directrix. DI is to FD as AE is to AF or AH (1. COR. 1.), and on account of the parallels, AE is to AH as BE is to HG. Therefore, ex æquo, DI is to FD, as BE is to HG. But DI is equal to BE, therefore FD is equal to HG.

LEMMA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LEMMA. 17.

If a cone VYy be cut by a plane through the axis VK, making the triangular section VYy, and also by two other planes ADa, 199, 200, 201.
GDg, each perpendicular to the plane VYy, of which planes one GDg is parallel to the base of the cone, and DB be the common section of the planes ADa, GDg, while Gg is the common section of the plane GDg with the plane VYy, the square of DB shall be equal to the rectangle GBg.

For it is demonstrated by every Writer on the Cone, that the section GDg is a circle, whose diameter is Gg; and therefore DB being drawn from a point in the circumference perpendicular to Gg (19. e. 11.), the square of DB will be equal to the rectangle GBg (35. & 3. e. 3.).

COR. 1. Hence if from any point D in the surface of the cone be drawn DB perpendicular to the plane VYy, and through B be drawn in the plane VYy the right line GBg parallel to Yy the common section of the plane VYy with the base of the cone, and meet the sides VY, Vy, in G, g; the square of DB shall be equal to the rectangle GBg.

For the plane GDg will be perpendicular to the plane VYy (18. e. 11.), and therefore will be a circular section, whose diameter is Gg. Also if any other plane passing through DB cut the cone, DB will be its common section with the plane GDg. Therefore the COR. is the same with the LEM.

COR. 2. If a plane ADa, not parallel to the base of the cone, cut the cone, making on its surface the line ADa, and from any points D therein be drawn the right lines DB perpendicular to Aa the common section of the planes ADa, VYy, or any right lines whatever be drawn on the plane ADa, and from the concourse B

FIG. 205, 206, 207. of the perpendiculars DB with Aa be drawn in the plane VYy the right lines GBg parallel to Yy; then if the plane ADa with all the lines drawn on it turn on Aa as an axis, 'till it coincide with the plane VYy, every thing will be exhibited on the plane VYy as it existed on the plane ADa, with the same relations to each other and to Aa, and the squares of the perpendiculars DB will be equal to the rectangles GBg, as on the plane ADa in its primary position.

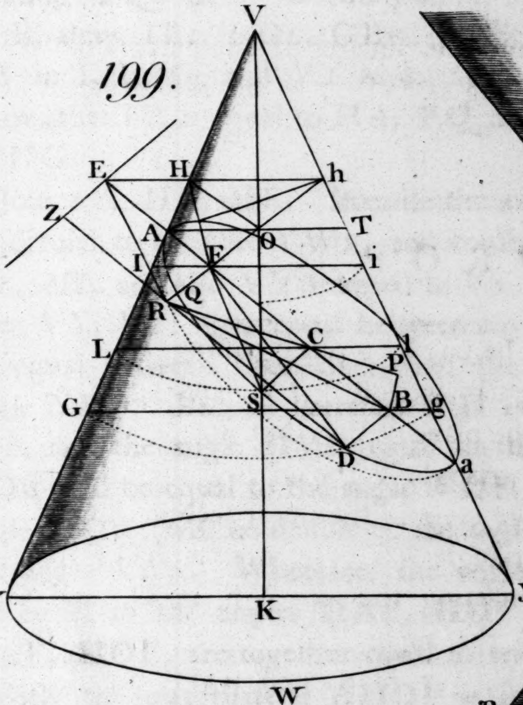
PROP. 54.

199, 200, 201. If a right cone YVy be cut by a plane passing through the axis VK, making the triangular section VYy, and again by a plane AQD perpendicular to the plane VYy, but not parallel to the base of the cone, cutting the triangle VYy in Aa their common section, and the surface of the cone in the linear section AQD. Also, if in the plane of the triangle VYy be drawn AOT parallel to the base of the cone, meeting the axis VK in O, and Oh be drawn making with VO the angle VOh equal to half the angle VAa, meeting Aa in F, Vy in h, and hHE parallel to AT be drawn, meeting Aa in E, and VY in H.

Then if through any point in the linear section AQD a plane parallel to the base of the cone be passed, cutting the sides of the triangle VYy; I say, that the distance of the point F from the point in the linear section shall be equal to the segment of the side VY or Vy, intercepted between Hh and the plane parallel to the base.

202, 203, 204. Suppose the plane AQD to turn on the common section Aa, as an axis, 'till it coincide with the plane VYy, exhibiting as in Cor. 2. of the preceding Lemma, the line AQD, which is formed on the surface of the cone by the section. Draw FQ perpendicular

199.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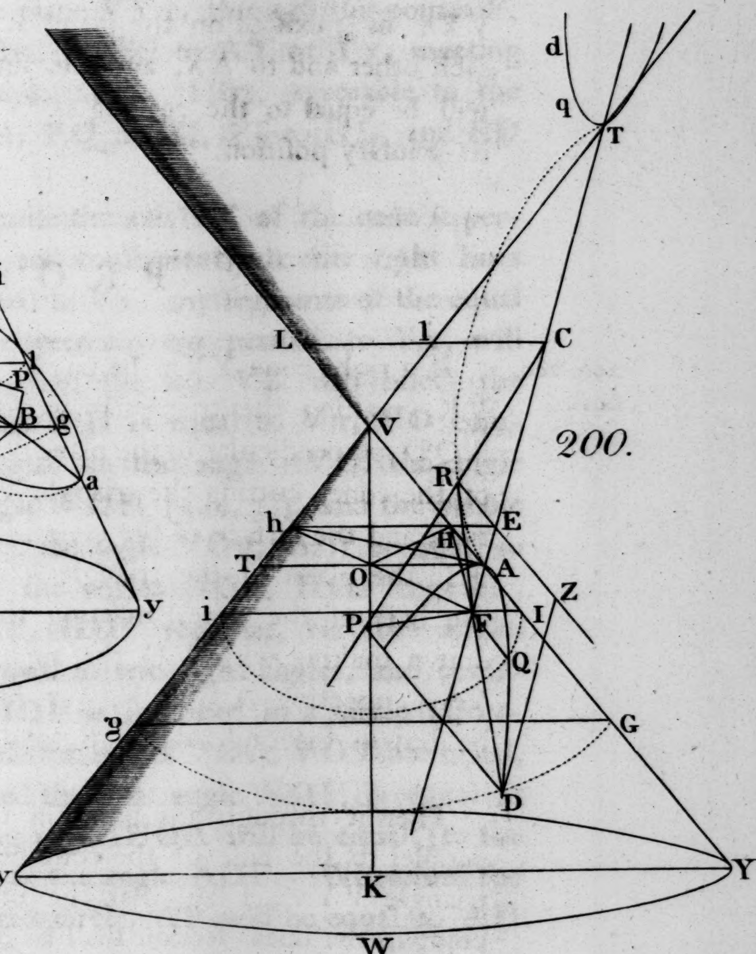
200.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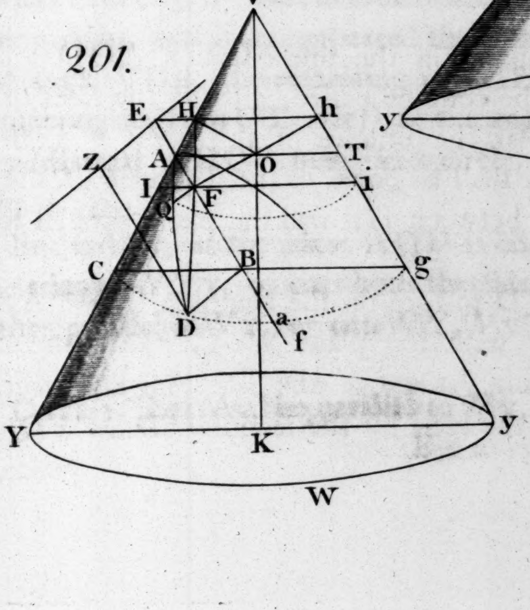
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pendicular to AA, meeting the linear section AQD in Q, and FIG. from any point D therein draw DB perpendicular to AA. Also when the plane AQD cuts both the sides VY, Vy, as in A, a, bisect Aa in C, and in FIG. 202. draw CS perpendicular to Aa, meeting AQD in S. In the plane VYy, through the points F, C, B, draw IFi, LCl, GBg, parallel to AT or Yy, meeting VY in I, L, G, and VY in i, l, g. I say, agreeable to the PROP. that FA is equal to HA, FQ to HI, FS to HL, and FD to HG.

Join hA, HO, HF. Because the axis VK of the cone is perpendicular to the base YWy, and consequently to the right lines Yy, AT, and also VY is equal to Vy, any segments of the equal sides VY, Vy, intercepted between any two parallels to Yy, will be equal between themselves, and the axis VK will bisect the angle YVy. Because therefore VH is equal to Vh, VO common, and the angle HVO equal to the angle hVO, the angle VOh will be equal to the angle VOH (4. e. 1.), and the whole angle HOh (will be double to the angle VOh, viz.) be equal to the angle VAa. Wherefore the angles HOh, HOF together, are equal to the angles HAF, HOF together, viz. the angles HAF, HOF, are together equal to two right angles, and consequently the quadrilateral AHOF is inscribed in a circle (CONVERSE. 22. e. 3.). But because the angles VOA, VOT are equal, being right, and it has appeared that the angle VOH is equal to the angle VOh, the remaining angle HOA will be equal (to the remaining angle hOT, viz.) to the angle AOF. Wherefore the quadrilateral AHOF being in a circle, AF will be equal to AH (29. e. 3.).

But farther, as the plane AQD is either parallel to a side Vy of the triangle VYy, or cuts both the sides VY, Vy; viz. as Aa is either parallel to Vy, or cuts VY, Vy in Aa, therefore

CASE 1. Let Aa be parallel to Vy.

 Because

FIG. 204. Because HOFA is in a circle, and AH is equal to AF, the angle AFH or EFH will be equal (to the angle AOF, viz.) to the angle EhF. Wherefore the triangles EFH, EhF, having also a common angle at E, are equiangular, and EH is to EF as EF is to Eh, viz. the square of EF is equal to the rectangle HEh. Moreover the square of FQ is equal to the rectangle IFi, as is also the square of BD to the rectangle GBg (LEM. 17.). But because the four right lines hE, hA, hF, hy, issue from the same point h, and AT parallel to one of them hE, and falling upon the other three, is bisected in the middle concourse O, these four right lines are harmonicals (LEM. 4.), and therefore EF parallel to one of them hy, and falling upon the other three, will be bisected in the middle concourse A. Wherefore EA being equal to AF, is equal to AH, and consequently EF is equal to HI, and EB to HG. Also because Aa is parallel to Vy, Eh, Fi, Bg, are equal between themselves, and on account of the parallels, AF is to AE (as IF is to EH, viz. as the rectangle IFi is to the rectangle HEh, viz.) as the square of FQ is to the square of EF. For the same reason is AB to AE or AF as the square of BD is to the square of EF. But AF has been shewn to be equal to AE, therefore the square of FQ is equal to the square of EF, and FQ is equal to EF, viz. to HI. In Ba towards the opposite parts of B make Bf equal to BF. Because Ff is double to FB, and EF is double to AF, the whole Ef is double to the whole AB, and Ef will be EF as AB is to AF. But it has appeared that AB is to AF as the square of BD is to the square of EF; therefore, ex æquo, the square of BD is to the square of EF (as Ef is to EF, viz.) as the rectangle FEf is to the square of EF. Therefore the square of BD is equal to the rectangle FEf. Add the common square of FB, and the square of FD will be equal to the square of EB (47. e. 1. & 6. e. 2.). Wherefore FD is equal to EB, viz. to HG.

CASE 2. When Aa cuts VY, Vy.

For

For the same reason as in the preceding case, the four right lines HE, HA, HF, Hy, are harmonicals, and the square of EF is equal to the rectangle HEh. Therefore Aa meeting these four harmonicals, is harmonically divided in the points of concourse E, A, F, a (1. COR. LEM. 4.). Round C with the distance of CA or Ca describe a circle meeting FQ in R, in FIG. 202., but in FIG. 203., meeting in R a perpendicular to Aa at the point E. Join ER, FR, CR. Because of the harmonic division, ER in FIG. 202. and FR in FIG. 203. touch the circles (3. COR. LEM. 10.), and consequently the rectangle AEa is equal to the square of ER. Also as before the rectangles IFi, LCl, GBg, are respectively equal to the squares of FQ, CS, BD (LEM. 17.). Because of the parallels, AB is to GB as AE is to EH, and aB is to gB as aE is to Eh; wherefore compounding these two ratios, the rectangle ABa is to the rectangle GBg as the rectangle AEa is to the rectangle HEh, viz. the rectangle ABa is to the square of BD as the square of ER is to the square of EF. Draw BP, Fp, in FIG. 202. perpendicular to CR, but in FIG. 203. let BD, FQ meet CR in P, p; and in FIG. 202. join BR, in FIG. 203. join FP. Because of the equiangular triangles, ER is to EF as CR is to FR or as CB is to BP. Therefore, ex æquo, the rectangle ABa is to the square of BD (as the square of CR is to the square of FR, or) as the square of CB is to the square of BP; and in FIG. 202. componendo, the rectangle ABa is to the square of BD as the square of AC is to the squares of BD, BP together. But it has appeared that the rectangle ABa is to the square of BD as the square of CR or AC is to the square of FR. Therefore the square of FR is equal to the squares of BD, BP. By the same reasoning it may be shewn that the square of FR is equal to the square of CS, and the square of FR equal to the squares of FQ, Fp, which indeed are only particular cases of the same general property. Because therefore the square of FR is equal to the squares of BD, BP; add the common square of FB, and the square FIG. 202, 203.

FIG. ſquare of BR will be equal to the ſquares of FD, BP. But the ſquare of BR is equal to the ſquares of RP, BP; therefore the ſquares of FD, BP together are equal to the ſquares of RP, BP together, and taking away the common ſquare of BP, the ſquare of FD will be equal to the ſquare of RP, and FD be equal to RP. For the ſame reaſon will FS be equal to RC, and FQ be equal to Rp, which as before are only particular caſes of a general property.

Alſo, in the caſe of Fig. 203. becauſe it has been ſhewn that the rectangle ABa is to the ſquare of BD as the ſquare of CR or AC is to the ſquare of FR, therefore componendo, the rectangle ABa is to the ſquare of BD as the ſquare of BC is to the ſquares of FR, BD together. But it has appeared that the rectangle ABa is to the ſquare of BD as the ſquare of BC is to the ſquare of BP. Therefore the ſquare of BP is equal to the ſquares of FR, BD together; and by the ſame reaſoning may it be ſhewn that the ſquare of Fp is equal to the ſquares of FR, FQ together, as being only another caſe of the ſame general property. But becauſe the ſquare of BP is equal to the ſquares of FR, BD together, add the common ſquare of BF, and the ſquare of FP will be equal to the ſquares of FD, FR together. But the ſquare of FP is equal to the ſquares of RP, FR together. Therefore the ſquares of FD, FR together are equal to the ſquares of RP, FR together, and taking away the common ſquare of FR, the ſquare of FD is equal to the ſquare of RP, and FD is equal to RP, in like manner as in the caſe of Fig. 202. For the ſame reaſon is FQ equal to RP, being as before a particular caſe of a general property.

In both caſes therefore, becauſe of the ſimilar triangles ERF, RFp, ER is to EF as RF is to Rp, and alternando, ER is to RF as EF is to Rp. But becauſe Aa is harmonically divided in E, F, and from E, F, are inflected ER, FR, to a point R in the circle ARa, ER is to RF as EA is to AF or AH (3. Cor. Lem. 6.). Therefore, ex æquo, EF is to Rp as EA is to AH,

viz.)

viz.) as EF is to HI. Wherefore Rp is equal to HI. But it has been ſhewn that FQ is equal to Rp, therefore FQ is equal to HI. FIG.

Laſtly, becauſe EA is to AH as ER is to RF, and on account of ſimilar triangles, ER is to RF as CF is to Cp, or as CB is to CP, and alſo EA is to AH as CF is to IL, and as CB is to LG (2. e. 6.); therefore ex æquo, CF is to Cp as CF is to IL, and CB is to CP as CB is to LG. Wherefore Cp is equal to IL, and CP is equal to LG. But it has appeared that Rp is equal to HI, wherefore the whole or remainder RC is equal to the whole or remainder HL, and conſequently the whole or remainder RP is equal to the whole or remainder HG. But it has been ſhewn that RC is equal to FS, and RP to FD, therefore FS is equal to HL, and FD to HG.

PROP. 55.

If a right cone be cut by a plane which is not parallel to the baſe, the linear ſection formed on the ſurface of the cone ſhall be a CONIC SECTION; and if the plane be parallel to a ſide of the cone, the ſection ſhall be a PARABOLA; if it meet both ſides of the cone, the ſection ſhall be an ELLIPSE; but if it cut the oppoſite cone alſo, which is the continuation of the former, the ſection ſhall be two OPPOSITE HYPERBOLAS, and the focus and directrix of each ſection ſhall be given.

Every thing remaining the ſame as in the laſt, I ſay, that AQD is a conic ſection, whoſe focus is F, and directrix the right line XX drawn through E perpendicular to Aa.

Draw

FIG. 202, 203, 204. Draw DZ perpendicular to XX. Because of the parallels, BE is to HG as FE is to HI. But BE is equal to DZ, and HG to FD (54.); therefore DZ is to FD as FE is to HI. Wherefore XX being a right line given in position, F a point without it, HI a right line given in magnitude, and the line AQD having this property, that the perpendicular distance of every point D therein from XX is to the distance of F from D, as the perpendicular distance of F from XX is to HI, the line AQD shall be a conic section (1. DEF.), XX will be its directrix (3. DEF.), F its focus (4. DEF.), and HI equal to the semi-parameter of the focal axis (5. DEF.), which axis is AA (6. DEF.).

204. If the plane AQD be parallel to a side of the cone, viz. if AA be parallel to a side VY of the triangle VYY, then FE is equal to HI, and therefore in this case AQD will be a parabola (2. DEF.).

202, 203. If the plane AQD cut the opposite sides of the cone, either towards the same or different parts of the vertex V, viz. if AA cut VY, VY, towards the same or different parts of V; because EC is to HL as EF is to HI, and it has been shewn in the preceding PROP. that RC or AC is equal to HL; therefore EC is to AC as EF is to HI. But as the plane cuts the opposite sides of the cone, viz. as AA cuts the sides VY, VY, towards the same or different parts of V, AC will be a part of EC, or EC will be a part of AC. Therefore in the first of these two cases, AQD will be an ellipse, in the latter the two opposite sections AQD, AQD, will be two opposite hyperbolas (2. DEF.).

SCHOL. If the circle meet ER again in R (FIG. 203.), and CR be joined, CR, CR will be the assymptotes of the hyperbolas, because FR, FR, touch the circle, which is described round the centre C with the distance of the semi-axis transverse, and FR is equal to the semi-axis secundus (VIDE 19. COR. DEF.).

202. COR. 1. If through the focus F of an ellipse be drawn FQ ordinately applied to the transverse axis AA, and meet the generating circle described round the centre C of the ellipse in R, and the diameter

'ameter CR be drawn. Then if from any point B in the axis be drawn BD parallel to the ordinates applied to A a, and meet the ellipse in D, and BP perpendicular to CR; I say that the squares of BD, BP together shall be equal to the square of the semi-axis secundus. FIG.

Every thing remaining the same as in this PROP. and in the preceding, it was shewn in the preceding that the square of FR is equal to the squares of BD, BP together, and that FR is equal to CS. But by this PROP. it appears that the line AQD is an ellipse, therefore CS is the semi-axis secundus, and consequently the squares of BD, BP together are equal to the square of CS.

COR. 2. The same things remaining as in the last COR. if FD be joined, I say that RP, the segment intercepted between R and the perpendicular BP shall be equal to FD.

It was shewn in the preceding PROP. that RP is equal to FD, and in this PROP. it is proved that the line AQD is an ellipse, therefore the COR. is true.

COR. 3. If through any point in the transverse axis of an hyperbola a right line be ordinately applied to the axis, and meet the section and an assymptote, the excess of the square of the segment intercepted between the axis and assymptote above the square of the semi-ordinate shall be equal to the square of the semi-axis secundus.

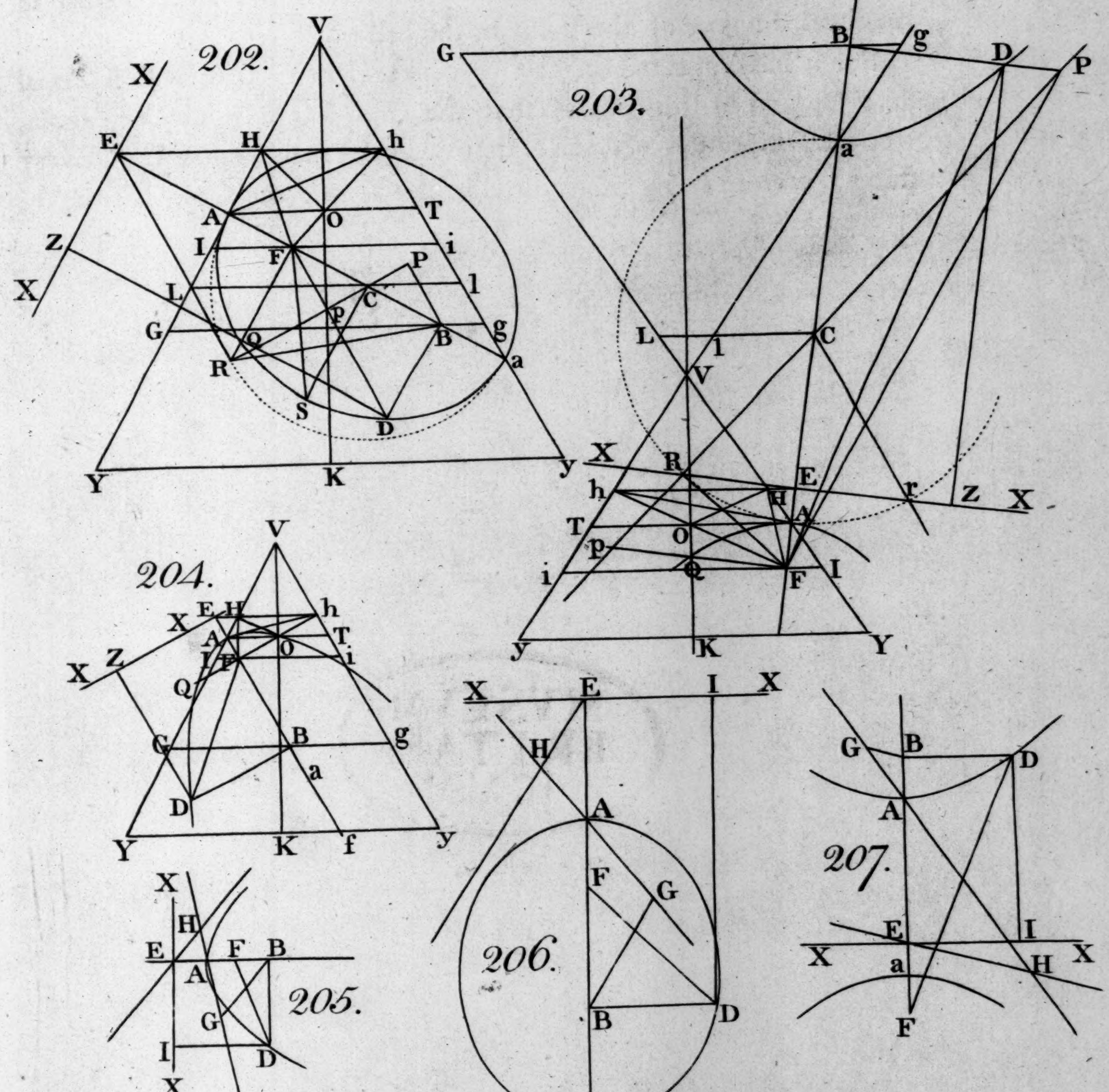
The same things remaining as in the PROP. I say that the excess of the square of BP above the square of BD shall be equal to the 203.
square of the semi-axis secundus.

For it has appeared in the preceding PROP. that in FIG. 203. the square of BP is equal to the squares of BD, FR together, and in this PROP. that the line AQD is an hyperbola, whose semi-axis secundus is equal to FR. Therefore the square of BP is equal to the square of BD together with the square of the semi-axis secundus, or the excess of the square of BP above the square of BD is equal to the square of the semi-axis secundus.

FIG. COR. 4. The ſame things remaining as in the laſt COR. if round the centre of the hyperbolas a circle be deſcribed with the diſtance of the ſemi-axis tranſverſe, and from any point in the ſection an ordinate be applied to the ſame axis, the ſegment of either aſſymptote intercepted between the circle and ordinate ſhall be equal to the focal diſtance of the point in the ſection.

For it has appeared in the preceding PROP. that RP is equal to FD, and in this PROP. that the line AQD is an hyperbola, whoſe aſſymptote is CR, axis tranſverſe AA, and centre C. Every thing therefore being as required in the COR., the COR. is true.

FINIS.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